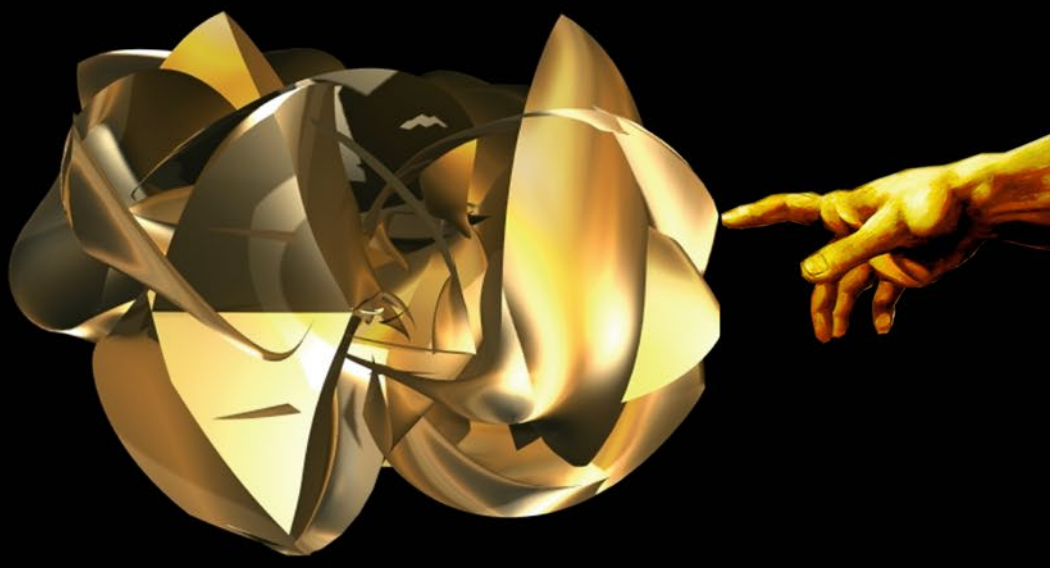
Computational Music Science



Guerino Mazzola

The Topos of Music IV: Roots

Appendices

Second Edition

Springer

Computational Music Science

Series editors

Guerino Mazzola
Moreno Andreatta

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The Topos of Music IV: Roots

Appendices

Second Edition

Springer

Guerino Mazzola
School of Music
University of Minnesota
Minneapolis, MN, USA

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Preface to the Second Edition

Comprendre, c'est
attraper le geste
et pouvoir continuer
Jean Cavaillès [181, p. 186]

A major reason for a second edition of *The Topos of Music*—besides the simple fact that the first edition is now sold out—goes back to spring 2002, when I was completing its first edition, published in fall 2002. I was asked to give a talk in the MaMuX seminar of the IRCAM in Paris, to explain how I applied the mathematics of *The Topos of Music* to my free jazz improvisations.

While preparing my talk I realized that despite the presence of mathematical music theory the decisive generator of my instant compositions was the gestural deployment of formulas, the "action painting" of musical thoughts, not the abstract formulas in their static facticity. First and foremost this was a shocking insight in view of the forthcoming publication of the formulaic setup in *The Topos of Music*.

Fortunately, I knew from Hermann Hesse that "every end is a beginning"[1], which meant in my case that the end of a scientific development as traced in the book's first edition initiated the next step: a music theory of gestures. It goes without saying that this new phase would not destroy the previous research, but incorporate it as the stratum of facticity in an extended ontology of embodiment, where facts are the output of processes and their gestural generators.

In the sequel, I discovered that I was far from being the first scholar and artist to discover the crucial role of gestures in music. For instance, free jazz pianist Cecil Tayor, music philosopher Theodor Wiesengrund Adorno, or lateral thinker Paul Valéry had clearly stressed the dancing essence of art, an insight that I had embodied in my own pianist's art, but never understood on an intellectual level.

Of course, I could not be satisfied by the very existence of gesture philosophy or gestural practice, just as I could not accept traditional music and performance theory when I started my enterprise of mathematical music theory in 1978. The gesturally colored thoughts and actions needed a rigorous conceptualization in the same vein as my efforts before the first edition of *The Topos of Music*.

In 2002, I was in the privileged position to work in the multimedia division of Peter Stucki at the Institut für Informatik of the University of Zurich. I had excellent PhD students, and we could, with one of them, Stefan Müller, realize a first experimental software for the gestural representation of a pianist's hand, a work presented at the ICMC conference in 2003 [772].

This experimental preliminary work was then taken as a point of departure for a mathematical theory of musical gestures. I presented this theory in a course in spring 2005 at École normale supérieure in Paris, a course that was later in 2007 taken as the material basis of my French book *La vérité de beau dans la musique* [718]. The first publication of a formally more evolved mathematical theory of musical gestures was

[1] Actually, he says that "Jeder Anfang ist ein Ende," but the reverse is immediate.

written with co-author Moreno Andreatta in 2007 [720]. This date could be called the birthday of a valid mathematical theory of musical gestures.

Until the publication of this second edition of *The Topos of Music*, several important conceptual extensions of the mathematical theory of musical gestures, models of musical gestural processes, as well as a number of theorems have been published. The decade from date of birth to the presence proved that the mathematical theory of musical gestures is an important added value to the theory described in the first edition of *The Topos of Music*.

We can however not state that this theory of gestures is in a complete state, quite the opposite is true: The coming years will reveal important news of theoretical as well as practical nature. So why did we make the decision to publish the present state of the art? The first argument is that the present state is rich enough to define concrete new directions, be it in music theory, such as harmony or counterpoint, be it in performance theory, or be it in the understanding of embodiment in the making of music. The second argument is that the present material, roughly 500 pages of new material, is ample enough to present a book's stature. And the third and very important argument is that we would like to communicate the state of the art in the spirit of Cavaillès: *Understanding is catching the gesture and being able to continue*. The co-authors of the gesture theory part, René Guitart, Jocelyn Ho, Alex Lubet, Maria Mannone, Matt Rahaim, and Florian Thalmann, are a wonderful confirmation of this philosophy. So let us continue!

Here is a summary of the new material and its authorship. Whenever I don't mention the author, it is my own contribution, all others are mentioned explicitly.

Until Part XIV, nearly everything is as in the book's first edition, refer to the preface of that edition (also included in this edition) for detailed summaries. The only new content—besides errata corrections—is Chapter 45 in Part XI, which is a shortened version of a paper [110] on a statistical analysis of Chopin's Prélude op. 28, No. 4, written with Jan Beran, Robert Goswitz, and Patrizio Mazzola.

Gesture theory starts with Part XV: Gesture Philosophy for Music. Chapter 57 gives an overview of philosophical aspects of gestures, including works by Jean-Claude Schmitt, Vilém Flusser, Michel Guérin, Adam Kendon, David McNeill, Juhani Pallasmaa, André Chastel, Émile Benveniste, Marie-Dominique Popelard, and Anthony Wall. In Chapter 58, we discuss the presemiotic approch to gestures in the French perspective of Maurice Merleau-Ponty, Gilles Deleuze, Jean Cavaillès, Charles Alunni, and Gilles Châtelet. Paul Valéry is also referenced in Section 59.4.

Chapter 59 deals with gestural aspects in cognitive science. After a review of Embodied AI and anthropology, Alex Lubet in Section 59.5 introduces us to gestural disability studies, focusing on two famous disabled pianists: Horace Parlan and Oscar Peterson (in his last years). Then in Section 59.6 Lubet reflects on perception of musical gesture as being inherently synaesthetic.

Chapter 60 concludes this part with a review of musical models of gesturality as proposed by Wolfgang Graeser, Theodor W. Adorno, Neil P. McAngus Todd, David Lewin, Robert Hatten, Marcelo Wanderley, Claude Cadoz, and Marc Leman.

Part XVI introduces the mathematics of gestures. Chapter 61 presents the mathematical concept of a gesture in a topological space and states the Diamond Conjecture, which deals with a hypothetical big space that unites algebraic and topological categories. Chapter 62 extends the theory from gestures in topological spaces to gestures in topological categories and introduces functorial gestures, i.e., functors on topological categories with values in the category of gestures, similar to functorial compositions in the previous theory.

Chapter 63 presents a generalized singular homology, where cubes are replaced by general hypergestures. Hypergesture homology applies to a gestural model of counterpoint and to a gestural refinement of performance stemma theory.

Chapter 64 presents—similar to Chapter 63—Stokes' Theorem for hypergestures. This theorem applies to problems in gestural modulation theory.

Chapters 65 and 66 discuss categories of local and global compositions, processes/networks, and gestures, together with their functorial relationships. This triple typology composition/process/gesture corresponds to the ontological dimension of embodiment with its three coordinates facts/processes/gestures.

In Sections 67.1–67.7 of Chapter 67, René Guitart develops a fascinating and demanding mathematical model of mathematical creativity, where thought is viewed as an algebra of gestures.

In Section 61.14, we, Maria Mannone and Guerino Mazzola, present a group-theoretical model of Georg Wilhelm Friedrich Hegel's initial discourse in his *Wissenschaft der Logik*, a model that applies to the Yoneda concept of creativity [726, Chapter 19.2]. We illustrate the method with an experimental composition by Mannone. This discussion extends over Sections 67.8–67.15 and completes Chapter 67.

Part XVII deals with concept architectures and software for musical gesture theory. Chapter 68 explains the denotator formalism for gestures over topological categories. Chapter 69 is a summary of the Java-based RUBATO® Composer software [739], written to sketch the framework of Chapters 70–74, where Florian Thalmann presents his gesture-oriented software component, the BigBang rubette. This discourse again follows the coordinates of the dimension of embodiment: facts, processes, gestures, which in this situation specify to: visualization and sonification of denotators (Chapter 71), BigBang's operation graph (Chapter 72), and gestural interaction and gesturalization (Chapter 73). In the final Chapter 74 of this part, Thalmann discusses musical examples.

Part XVIII is entitled *The Multiverse Perspective* because it opens up the relationship of gesture theory with string theory in theoretical physics. After a critical review of Hermann Hesse's *Glasperlenspiel* with regard to its gestural deficiencies, we, Mazzola and Mannone, develop the Euler-Lagrange formalism of world-sheets for musical gestures. This theory extends to functorial global gestures over global topological categories.

Part XIX is dedicated to applications of gesture theory to a number of musical themes.

Chapter 79 deals with singular gesture homology being applied to counterpoint.

Chapter 80 introduces a gestural restatement of modulation theory, applying in particular Stokes' Theorem for hypergestures.

Chapter 81 applies gesture theory to a gestural performance stemma theory.

Chapter 82 is written by Jocelyn Ho as a creative presentation of composition and analysis as embodied gestures in an inter-corporeal world. She presents two compositions, Toru Takemitsu's *Rain Tree Sketch II* and her own composition *Sheng* for piano, smartphones, and fixed playback.

Chapter 83 is Mannone's analysis and classification of a conductor's movements from the viewpoint of gesture theory.

Chapter 84 is a review of gestural aspects that were developed in *Flow, Gesture, and Spaces in Free Jazz* [721].

Chapter 85 is written by Matt Rahaim and presents the gestural approach to understanding Hindustani music in its vocal gesturality.

Chapter 86 is a first approach, written by Mannone, to a future theory of vocal gestures. The short addendum was written by Mazzola.

The Appendix has been enriched by additional complements on mathematics (Chapter J) plus complements on physics (Chapter K).

The Leitfaden III has been added to the original Leitfaden I & II for the gestural chapters.

The ToM_CD has been updated, containing now the present book's pdf ToposOfMusic.pdf. However the original CD is no longer added to the book, instead the ToM_CD can be downloaded from

www.encyclospace.org/special/ToM_CD.zip.

Concerning the division of the now very large book into parts, this is the split:

- Volume I: *Theory*, Prefaces and Table of Contents, Parts I to VII
- Volume II: *Performance*, Parts VIII to XIV
- Volume III: *Gestures*, Parts XV to XIX
- Volume IV: *Roots*, Appendices

My sincere acknowledgments go to my co-authors and to Springer's Ronan Nugent and Frank Holzwarth as well as to Birkhäuser's Thomas Hempfling.

Minneapolis, October 2017 Guerino Mazzola

Preface

Man kann
einen jeden Begriff,
einen jeden Titel,
darunter viele Erkenntnisse gehören,
einen logischen Ort nennen.
Immanuel Kant [519, p. B 324]

This book's title subject, *The Topos of Music*, has been chosen to communicate a double message: First, the Greek word "topos" ($\tau\acute{o}\pi o\varsigma$ = location, site) alludes to the logical and transcendental location of the concept of music in the sense of Aristotle's [40, 1154] and Kant's [519, p. B 324] topic. This view deals with the question of *where music is situated* ***as*** *a concept*—and hence with the underlying ontological problem: *What is the type of being and existence of music?* The second message is a more technical understanding insofar as the system of musical signs can be associated with the mathematical theory of *topoi*, which realizes a powerful synthesis of geometric and logical theories. It laid the foundation of a thorough geometrization of logic and has been successful in central issues of algebraic geometry (Grothendieck, Deligne), independence proofs and intuitionistic logic (Cohen, Lawvere, Kripke).

But this second message is intimately entwined with the first since the present concept framework of the musical sign system is technically based on topos theory, so the topos of music receives its topos-theoretic foundation. In this perspective, the double message of the book's title in fact condenses to a unified intention: to unite philosophical insight with mathematical explicitness.

According to Birkhäuser's initial plan in 1996, this book was first conceived as an English translation of my former book *Geometrie der Töne* [682], since the German original had suffered from its restricted access to the international public. However, the scientific progress since 1989, when it was written, has been considerable in theory and technology. We have known new subjects, such as the denotator concept framework, performance theory, and new software platforms for composition, analysis, and performance, such as RUBATO® or OpenMusic. Modeling concepts via the denotator approach in fact results from an intense collaboration of mathematicians and computer scientists in the object-oriented programming paradigm and supported by several international research grants.

Also, the scientific acceptance of mathematical music theory has grown since its beginnings in the late 1970s. As the first acceptance of mathematical music theory was testified to by von Karajan's legendary Ostersymposium "Musik und Mathematik" in 1984 in Salzburg [383], so is the significantly improved present status of acceptance testified to by the Fourth Diderot Forum on Mathematics and Music [711] in Paris, Vienna, and Lisbon 1999, which was organized by the European Mathematical Society. The corresponding extension of collaborative efforts in particular entail the inclusion of works by other research groups in this

book, such as the "American Set Theory", the Swedish school of performance research at Stockholm's KTH, or the research on computer-aided composition at the IRCAM in Paris.

Therefore, as a result of these revised conditions, *The Topos of Music* appears as a vastly extended English update of the original work. The extension is visibly traced in the following parts which are new with respect to [682]: Part II exposes the theory of denotators and forms, part V introduces the topological theories of rhythms and motives, part VIII introduces the structure theory of performance, part IX deals with the expressive semantics of performance in the language of performance operators and stemmata (genealogical trees of successively refined performance), part X is devoted to the description of the RUBATO® software platform for representation, analysis, composition, and performance, part XI presents a statistical analysis of musical analysis, part XII concludes the subject of performance with an inverse performance theory, in fact a first formalization of the problem of music criticism.

This does however not mean that the other parts are just translations of the German text. Considerable progress has been made in most fields, except the last part XIV which reproduces the status quo in [682]. In particular, the local and global theories have been thoroughly functorialized and thereby introduce an ontological depth and variability of concepts, techniques, and results, which by far transcend the semiotically naive geometric approach in [682]. The present theory is as different from the traditional geometric conceptualization as is Grothendieck's topos theoretic algebraic geometry from classical algebraic geometry in the spirit of Segre, van der Waerden, or Zariski.

Beyond this topos-theoretic generalization, the denotator language also introduces a fairly exceptional technique of circular concept constructions. This more precisely is rooted in Finsler's pioneering work in foundations of set theory [322], a thread which has been rediscovered in modern theoretical computer sciences [5]. The present state of denotator theory rightly could be termed a Galois theory of concepts in the sense that circular definitions of concepts play the role of conceptual equations (corresponding to algebraic equations in algebraic Galois theory), the solutions of which are concepts instead of algebraic numbers.

Accordingly, the mathematical apparatus has been vastly extended, not only in the field of topos theory and its intuitionistic logic, but also with regard to general and algebraic topology, ordinary and partial differential equations, Pólya theory, statistics, multiaffine algebra and functorial algebraic geometry. It is mandatory that these technicalities had to be placed in a more elaborate semiotic perspective. However, this book does not cover the full range of music semiotics, for which the reader is referred to [703]. Of course, such an extension on the technical level has consequences for the readability of the theory. In view of the present volume of over 1300 pages, we could however not even make the attempt to approach a non-technical presentation. This subject is left to subsequent efforts. The critical reader may put the question whether music is really that complex. The answer is yes, and the reason is straightforward: We cannot pretend that Bach, Haydn, Mozart, or Beethoven, just to name some of the most prominent composers, are outstanding geniuses and have elaborated masterworks of eternal value, without trying to understand such singular creations with adequate tools, and this means: of adequate depth and power. After all, understanding God's 'composition', the material universe, cannot be approached without the most sophisticated tools as they have been elaborated in physics, chemistry, and molecular biology.

So who is recommended to read this book? A first category of readers is evidently the working scientist in the fields of mathematical music theory, the soft- and hardware engineer in music informatics, but also the mathematician who is interested in new applications from the above fields of pure mathematics. A second category are those theoretical mathematicians or computer scientists interested in the Galois theory of concepts; they may discover interesting unsolved problems. A third category of potential readers are all those who really want to get an idea of what music is about, of how one may conceptualize and turn into language the "ineffable" in music for the common language. Those who insist on the dogma that precision and beauty contradict each other, and that mathematics only produces tautologies and therefore must fail when aiming at substantial knowledge, should not read such a book.

Despite the technical character of *The Topos of Music*, there are at least four different approaches to its reading. To begin with, one may read it as a philosophical text, concentrating on the qualitative passages, surfing over technical portions and leaving those paragraphs to others. One may also take the book as a dictionary for computational musicology, including its concept framework and the lists of musical objects

and processes (such as modulation degrees, contrapuntal steps) in the appendices. Observe however, that not all existing important lists have been included. For example, the list of all-interval series and the list of self-addressed chords are omitted, the reader may find these lists in other publications. Thirdly, the working scientist will have to read the full-fledged technicalities. And last, but not least, one may take the book as a source for ideas of how to go on with the whole subject of music. The GPL (General Public License[2]) software sources in the appended CD-ROM may support further development.

The prerequisites to a more in-depth reading of this book are these. Generally speaking, a good acquaintance with formal reasoning as mathematics (including formal logic) preconizes, is a conditio sine qua non. As to musicology and music theory, the familiarity with elementary concepts, like chords, motives, rhythm, and also musical notation, as well as a real interest in understanding music and not simply (ab)using it, are recommended. For the more computer-oriented passages, familiarity with the paradigm of object-oriented programming is profitable. We have not included the appendix on mathematical basics because it should help the reader get familiar with mathematics, but as an orientation in fields where the specialized mathematician possibly needs a specification of concepts and notation. The appendix was also included to expose the spectrum of mathematics which is needed to tackle the formal problems of computational musicology. It is by no means an overkill of mathematization: We have even omitted some non-trivial fields, such as statistics or Lambda calculus, for which we have to apologize.

There are different supporting instances to facilitate orientation in this book. To begin with, the table of contents and an extensive subject and name index may help find one's key-words. Further, following the list of contents, a leitfaden (on page xlv) is included for a generic navigation. Each chapter and section is headed by a summary that offers a first orientation about specific contents. Finally, the book is also available as a file `ToposOfMusic.pdf` with bookmarks and active cross-references in the appended CD-ROM (see page xlvii for its contents). This version is also attractive because the figures' colors are visible only in this version.

In order to obtain a consistent first reading, we recommend chapters 1 to 5, and then appendix A: Common Parameter Spaces (appendix B is not mandatory here, though it gives a good and not so technical overview of auditory physiology). After that, the reader may go on with chapter 6 on denotators and then follow the outline of the leitfaden (see page xlv).

This book could not have been realized without the engaged support of nineteen collaborators and contributors. Above all, my PhD students Stefan Göller and Stefan Müller at the MultiMedia Laboratory of the Department of Information Technology at the University of Zurich have collaborated in the production of this book on the levels of the LaTeX installation, the final production of hundreds of figures, and the contributions sections 20.2 through 20.5 (Göller) and sections 47.3 through 47.3.6.2 (Müller). My special gratitude goes to their truly collaborative spirit.

Contributions to this book have been delivered by (in alphabetic order): By Carlos Agon, and Gérard Assayag (both IRCAM) with their precious Lambda-calculus-oriented presentation of the object-oriented programming principles in the composition software OpenMusic described in chapter 52, Moreno Andreatta (IRCAM) with an elucidating discourse on the American Set Theory in section 11.5.2 and section 16.3, Jan Beran (Universität Konstanz) with his contribution to the compositional strategies in his original composition [103] in section 11.5.1.1, as well as with his inspiring work on statistics as reported in chapters 43 and 44, Chantal Buteau (Universität and ETH Zürich) with her detailed review of chapter 22, Roberto Ferretti (ETH Zürich) with his progressive contributions to the algebraic geometry of inverse performance theory in sections 39.8 and 47.2, Anja Fleischer (Technische Universität Berlin) with her short but critical preliminaries in chapter 23, Harald Fripertinger (Universität Graz) with his 'killer' formulas concerning enumeration of finite local and global compositions in sections 11.4, 16.2.2 and appendix C.3.6, Jörg Garbers (Technische Universität Berlin) with his portation of the RUBATO® application to Mac OS X, as documented in the screenshots in chapters 40, 41, Werner Hemmert (Infineon) with a very up-to-date presentation of room acoustics in section A.1.1.1 and auditory physiology in appendix B.1 (we would have loved to include more of his knowledge), Michael Leyton (DIMACS, Rutgers University) with a formidable cover figure entitled "Dark Theory", a beautiful subtitle to this book, as well as with innumerable discussions around time and its reduction to symmetries as presented in chapter 48, Emilio Lluis Puebla (UNAM, Mexico City)

[2] A legal matter file is contained in the book's CD-ROM, see page xlvii.

with his unique and engaged promotion and dissipation of mathematical music theory on the American continent, especially also in the preparation and critical review of this book, Mariana Montiel Hernandez (UNAM, Mexico City) with her critical review of the theory of circular forms and denotators in section 6.5 and appendix G.2.2.1, Thomas Noll (Technische Universität Berlin) with his substantial contributions to the functorial theory of compositions, and for his revolutionary rebuilding of Riemann's harmony and its relations to counterpoint, Joachim Stange-Elbe (Universität Osnabrück) with a very clear and innovative description of his outstanding RUBATO® performance of Bach's contrapunctus III in the *Art of Fugue* in sections 42.2 through 42.4.3, Hans Straub with his adventurous extensions of classical cadence theory in section 26.2.2 and his classification of four-element motives in appendix O.4, and, last but not least, Oliver Zahorka (Out Media Design), my former collaborator and chief programmer of the NeXT RUBATO® application, which has contributed so much to the success of the Zürich school of performance theory. To all of them, I owe my deepest gratitude and recognition for their sweat and tears.

My sincere acknowledgments go to Alexander Grothendieck, whose encouraging letters and, no doubt, awe inspiring revolution in mathematical thinking has given me so much in isolated phases of this enterprise. My acknowledgments also go to my engaged mentor Peter Stucki, director of the MultiMedia Laboratory of the Department of Information Technology at the University of Zurich; without his support, this book would have seen its birthday years later, if ever. My thanks also go to my brother Silvio, who once again (he did it already for my first book [670]) supported the final review efforts by an ideal environment in his villa in Vulpera. My thanks also go to the unbureaucratic management of the book's production by Birkhäuser's lector Thomas Hempfling and the very patient copy editor Edwin Beschler. All these beautiful supports would have failed without my wife Christina's infinite understanding and vital environment—if this book is a trace of humanity, it is also, and strongly, hers.

Vulpera, June 2002 Guerino Mazzola

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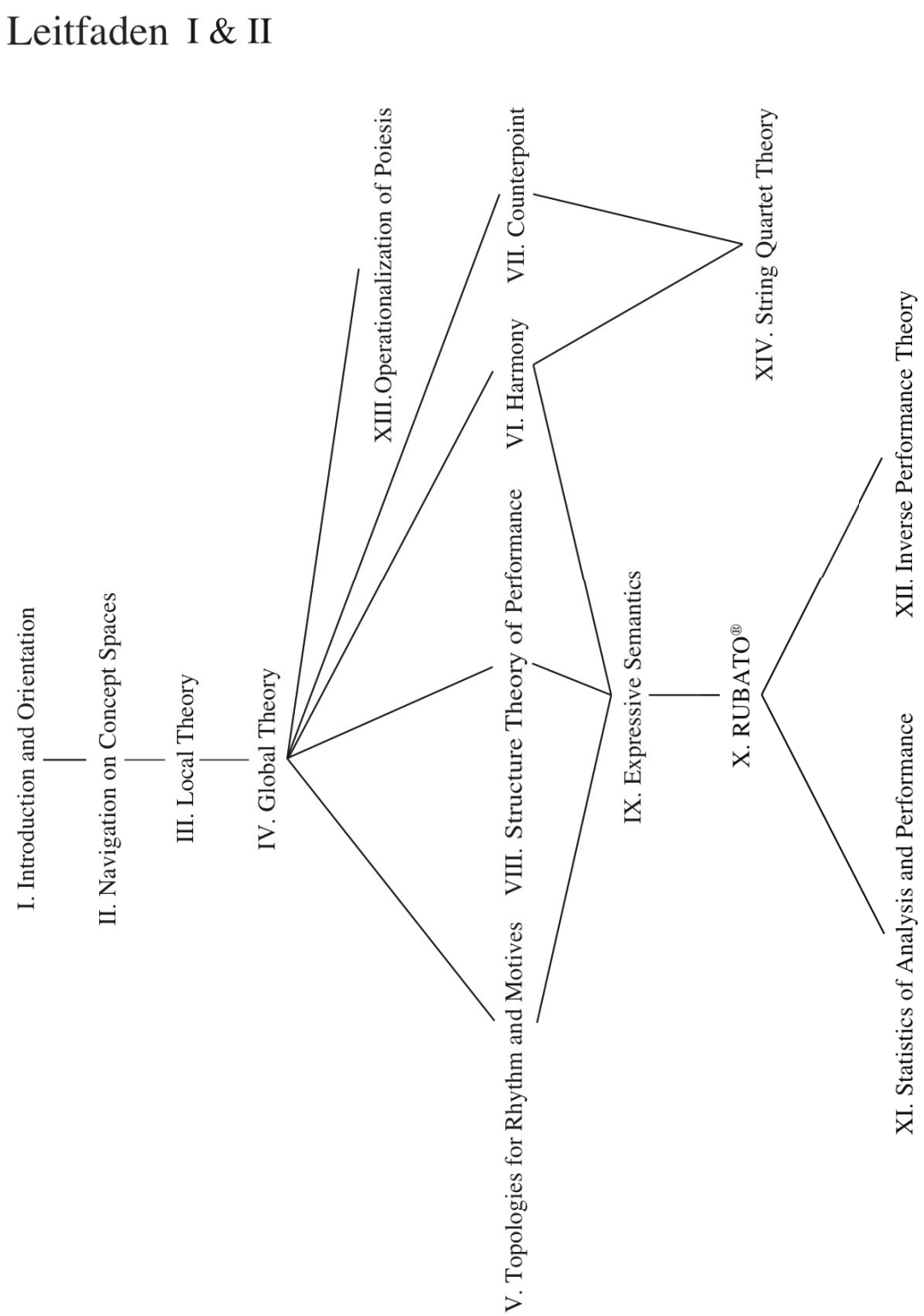
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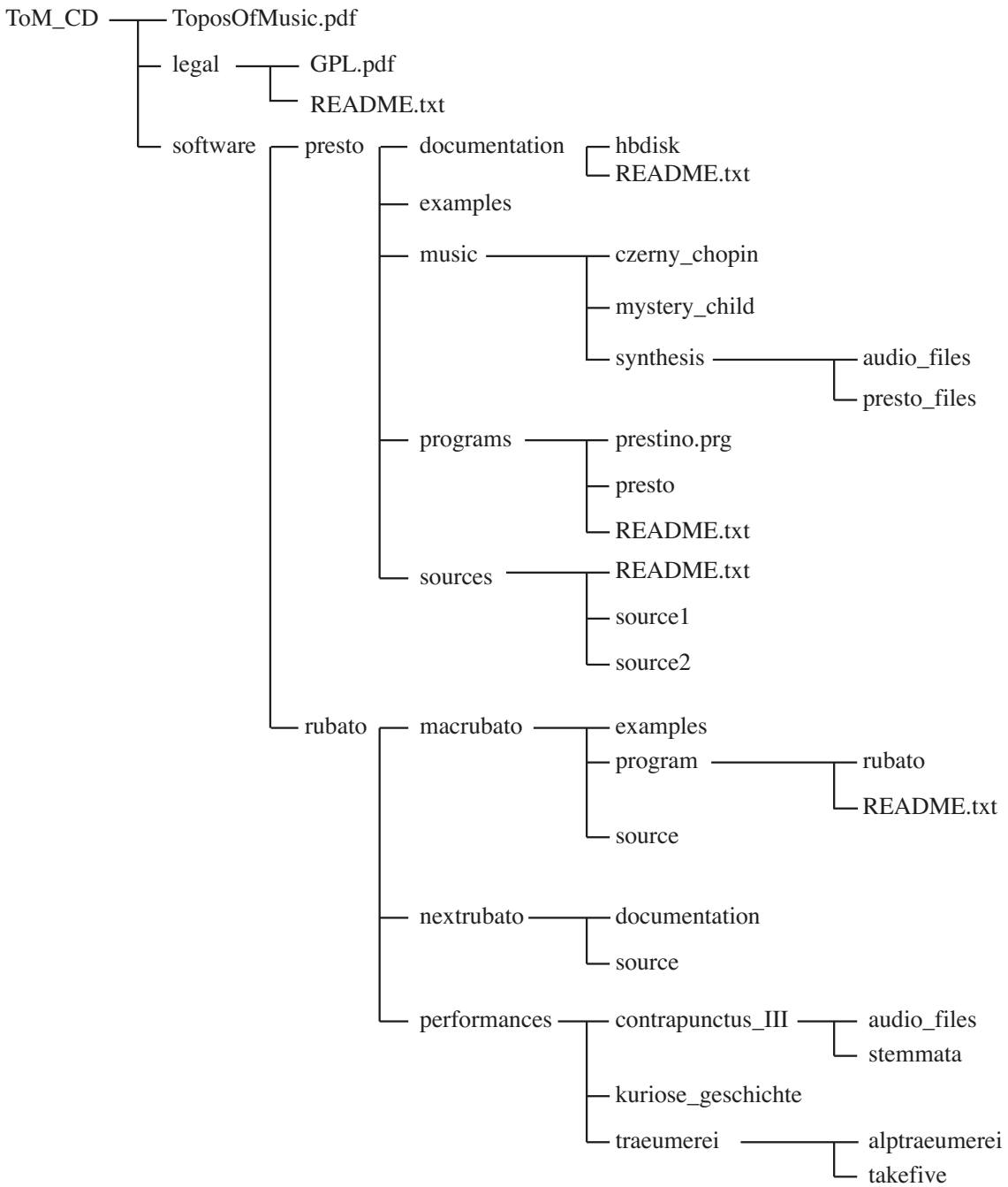
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Part XX

Appendix: Sound

A

Common Parameter Spaces

This appendix chapter is an overview, not an exhaustive treatise of spaces which parametrize sound objects. These spaces where sounds are positioned always define an aspect, never the totality of music thinking, and every attempt to define a preferred space will narrow the music thinking, not the music. The best that can occur is that we offer an encompassing or at least a representative ensemble of parameter spaces which are interrelated by a precise relation. To this end, it is recommended to distinguish topographic positions, above all in their realities and communicative perspectives. This will also entail the corresponding mathematics.

We start by the physical descriptions, turn over to more mathematical abstractions and the describe more symbolic viewpoints which we call interpretative since they are not just a new mathematical rephrasing of a priori equivalent physical description, but express abstraction with some mental background constructions.

A.1 Physical Spaces

As a physical object, a *sound*[1] is a more or less regular variation of normal air pressure[2] as a function of time. Starting at a determined *onset time e sec*, it starts from a source Q at position $q = q_Q\ m$ in the ordinary physical space and expands as a wave. At a location x and time t, the *pressure variation* (the difference from the normal pressure) $p_x^Q(t)\ Nm^{-2}$ is perceived as a longitudinal air wave, i.e., with a pressure front perpendicular to the waves expanding direction, see Figure A.1.

For a punctual sound source, however, the wave front at $x \neq q$ is a spheric surface; we can write

$$p_x^Q(t) = |x-q|^{-1} p_q(t - |x-q|/v), \tag{A.1}$$

where v is the expansion velocity of the wave[3]. It is calculated by the formula $v = C.\sqrt{T}$, where $T\ K$ is the absolute temperature in Kelvin degrees, and C is a constant with value $C = 20.1$ for the normal pressure. For normal conditions, we have $v \approx 343\ msec^{-1}$. But with complex sound sources and room-specific reflection and refractions, the pressure variation of a sound may be an overlapping of different spheric wave components. If several sound sources $Q_1, \ldots Q_s$ are given, the resulting pressure variation sums up to $p_x^{Q_\cdot}(t) = \sum_{1 \leqslant i \leqslant s} p_x^{Q_i}(t)$, and it is the integral $\int dp_x^{dQ}(t)$ of a family of infinitesimal point sources dQ. In general, from the knowledge of $p_x^{Q_\cdot}(t)$, one cannot infer the original source functions. This is like with painting where in general one perspectivic image does not allow us to infer the original object configuration: The esthesic position is not sufficient to reconstruct poiesis[4].

[1] German: "Klang".

[2] At the zero height above sea and zero degrees Celsius, this is $\approx 1.1013.10^5 Nm^{-2}$.

[3] It is known [914] that the square of the pression variation is proportional to the intensity, i.e., energy flow per surface and time unit, and the latter, by energy conservation, decreases proportionally to the square of the distance $|x-q|$, whence the formula.

[4] See chapter 2 for the concepts of music topography, such as "esthesic", "neutral", or "poietic".

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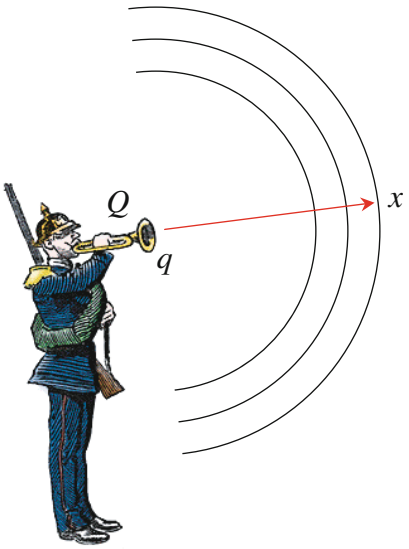

Fig. A.1. The prototypical punctual sound source and the spherical sound wave.

The fundamental problem of musical acoustics is that the neutral data, the objectively measurable pressure values, are far from what is intended by musicians, i.e., the neutral data is not what is interesting and what is the message. So one of the most important tasks of musical acoustics is the interpretation of the neutral data, in other words, of what is behind the data, what could be the hidden parameters of the audible phenomenena. And it is one of the worst tragedies of traditional music-acoustics and psycho-acoustics that the fundamental difference of neutral and poietic levels is ignored and disregarded.

A.1.1 Neutral Data

In order to describe the neutral sound data, let us first concentrate on the information at the source location $q = q_Q$ of a point source Q, in an idealized model of a single instrument. We shall come back to room acoustics in the next Section A.1.1.1. The source sound (variation) event $p_q(t)$ is usually a finite event, starting at time e, and ending after the duration d. The variation between these time limits is also limited by the maximal amplitude Am of the total pressure variation. So the function $p_q(t)$ is the affine image of a normalized function $p_q^0(t)$ which starts at time $e = 0$, has duration $d = 1$, and amplitude $A = 1$. More precisely:

$$p_q(t) = Support(A, e, d)(p_q^0)(t) = A.p_q^0((t - e)/d). \tag{A.2}$$

The operator $Support(A, e, d)$ reduces the unknown sound event to a normalized event. What happens between the normalized unit supports is however completely arbitrary. It may be a percussive sound or the sound of a Stradivari violin. The normalization by the support operator is completely harmless, but not much more than this data can be traced on the neutral level. The theory of all the rest is far from neutral; we are going to deal with this in Section A.1.2.

A.1.1.1 Room Acoustics

The room is an important part in the information chain from the information source (instrument, speaker, public address system) to the receiver (listener, director, artist). Room sizes vary from small living rooms to huge cathedrals or concert halls. In this section, we want to tackle only basic features, reverberation time and acoustical power. For a large, irregular room, we can visualize the acoustical conditions by imagining a

wave traveling inside the room. This wave travels in a straight line until it strikes a surface. It is reflected off the surface at an angle equal to the angle of incidence and travels in this direction until it strikes another surface. Because sound travels about 343 $msec^{-1}$, many reflections will occur within a small time span.

Absorption. After a wave has undergone a reflection from a wall that is absorbing, its intensity will be less during its next traverse of the room. In a large, irregular room, the number of waves traveling are so numerous that at each surface all directions of incident flow are equally probable. The sound absorption coefficient α is therefore taken to be averaged for all angles of incidence. All materials have absorption coefficients that are different at different frequencies[5] [111]. In the frequency range of interest between 250 Hz and 4 kHz, plain walls and floors, as well as closed windows, have absorption indices below 0.2. Higher absorption can be achieved with acoustic tiles and, at least in the upper frequency range, with thick carpets and draperies. The total absorption A of a room is the sum of the product of surfaces A_i m^2 and absorption coefficients α_i: $A = \Sigma_i \alpha_i A_i$.

If there is an open window in the room, all the energy incident on its area will pass outdoors and none will be reflected.[6] The absorption of an area of acoustical material in a room can therefore be expressed in terms of the equivalent area of an open window. For this reason, the total absorption A of a room can be characterized by its equivalent "open window surface".

Critical Distance. The sound field which builds up in a room is fundamentally different from the free field situation. Let us first assume a sound source in a free field, i.e., a loudspeaker on a high post emitting sound in all directions. If the sound radiation is unidirectional and no reflections occur, the source will emit a spherical wave and sound pressure will decrease inversely proportional with the distance from the source.

For loss-less reflecting walls on all sides around the source, the sound waves will be reflected over and over. In the case of no absorption in the air, the sound pressure will now rise and rise, as no energy loss occurs. Small absorption will cause an equilibrium. However, the sound field in the room will no longer be a directional spherical field because the reflected waves will by far dominate over the direct sound wave. The sound field will be diffuse, reflections will arrive from all directions with equal probability. Only in the close vicinity of the small sound source is the sound field directional, because there the sound pressure of the unreflected direct sound wave will dominate over the diffuse sound field. In summary, the sound field in a room will be directional only close to the sound source Q. There, as we know from the above, at point x, the sound pressure falls with $1/|x-q|$. The sound field at a large distance from the source is diffuse. Sound pressure is almost constant and much higher compared to a free sound field. The distance from the source, where the transition between these distinct regions, occurs is called *critical distance* or *diffuse field distance* r_H. If the average absorption $\bar{\alpha} \leqslant 0.4$, the critical distance for the absorption A can be calculated with a precision of about 10% [111]:

$$r_H = \sqrt{A/16\pi} \approx 0.14\sqrt{A}. \tag{A.3}$$

By the inverse distance law, the sound *pressure decrease factor* ρ_D in the diffuse field of a room can be estimated by the formula:

$$\rho_D = 1/r_H \approx \frac{7.1}{\sqrt{A}}. \tag{A.4}$$

Reverberation Time. Temporal effects during the onset and the decay of sound are essential features in rooms and are not present in a free sound field. In large rooms, such as a cathedral, long decay times are apparent.The human auditory system can distinguish whether a sound source is located in a large room, a small room or even whether it is not in a room at all, but outside. The human ear is obviously able to extract information about the room size from the temporal structure of sound. We will therefore discuss the onset and decay of sound.

[5] See Section A.1.2 for the discussion of the frequency concept.

[6] This statement is strictly true only if the window is several wavelengths wide and high, otherwise diffraction will occur.

When the sound leaves its source, the direct sound reaches the receiver first, its delay is determined by the distance sound has to travel divided by the velocity of sound. With further delay, reflected sound waves with only a slightly longer travel distance arrive, and later, also reflections with longer routes and multiple reflections arrive. The sound pressure is increasing until it reaches its steady state value. Only the direct sound gives information about the location of the source, which is exploited by our hearing system for sound localization. The build-up of the reverberation is only perceived in highly reverberant rooms. The reverberation is much more perceptible after the source is muted, and the echo may be still noticeable after seconds. The direct sound ceases after the propagation time from the source to the receiver, however, all the reflected sound waves still arrive. Their intensity will be reduced by factor $1-\bar{\alpha}$ after each reflection on a wall. Therefore, the sound pressure will decrease exponentially.
The *reverberation time* T is defined as the time required for the sound to decay by a sound pressure level[7] of 60 dB. Let α_L m^{-1} be the absorption in air (a highly frequency- and humidity-dependent variable), and let V m^3 be the volume of the room. Then the reverberation time T can be estimated with *Sabine's* formula [1158]:

$$T \approx \frac{0.161V}{A + 0.46\alpha_L V} \; sec. \tag{A.5}$$

In the frequency range below 4 kHz, sound absorption in air is usually neglected, whereas for frequencies above 4 kHz, the reverberation time T is mainly determined by the absorption in air α_L.
The reverberation time is the most important variable to describe the acoustics of rooms. In rooms with a long reverberation time, sources with a relatively low level yield a high sound intensity, however, speech intelligibility is decreased due to increased temporal masking. As a compromise, the reverberation time must be "appropriate" to the room size. For speech, reverberation time should be between 0.5 *sec* and 1 *sec* (increasing with room size), for music presentations 1 *sec* to 2 *sec* are acceptable.
Beyond the global description of a room by the reverberation time, the temporal fine-structure of the reverberation, the temporal incidence of the reflections, is of interest. From the first reflections, the human hearing system is able to extract information about the size of the room. If the first reflections occur very early (1 *msec* to 10 *msec* after the direct sound wave), the sound color—especially for music recordings—is altered. Reflections in the time span from 10 *msec* to 50 *msec* increase the perceived loudness. Single echoes, arriving with a delay of more than about 100 *msec*, are perceived as echoes. Very disturbing are periodic echoes, which are generated between parallel walls, for example. For good room acoustics, the reflections should be homogeneous and the intensity should decrease with time. Single echoes should not be larger than 5 dB compared with their temporal vicinity. In concert halls, the reverberation time is measured as a function of frequency and additional sound absorbers and reflectors are placed to improve the acoustics.

Remark 29 This sketchy discussion shows that room acoustics is a complex topic, therefore acoustic experts should be consulted already in the planning of music rooms. This need is documented by plenty of examples, where the acoustics of rooms built for the purpose of audio presentations is so bad that speech intelligibility is severely hampered. Thoroughly planned room acoustics is essential not only for concert and lecture halls, but also for most other rooms such as offices and even hallways and production areas, to keep noise levels down and achieve an environment which is pleasing to the ear.

Remark 30 We have also included this discussion since it makes plausible that there is no chance to integrate a poor theory of room acoustics in a valid music-theoretic framework, and that great efforts should be made to lift this status in order to give the compositions with room acoustical specifications a firm background.

A.1.2 Sound Analysis and Synthesis

This central subject of acoustics relates to the poietic (synthesis) and esthesic (analysis) aspects of neutral sound data, more specifically, of the sound pressure variation function $p(t) = p_q^0(t)$ at a source location q

[7] See Section A.2.2 for the definition of loudness.

and cast in a determined standard support (see Section A.1.1). More precisely, we are considering a time function p which is defined for all times, vanishes outside a finite time interval, e.g., $[0, 1]$, and has a finite absolute amplitude supremum $sup(|p|)$, e.g., $sup(|p|) = 1$.

Sound synthesis means that we have exhibited an operator σ which for each sequence of its finite or infinite number of numeric (mostly real-valued) parameters $x_1, x_2, \ldots$ yields a function $p = \sigma(x_1, x_2, \ldots)$ of the described type. Analysis then means that we are given p and the operator σ and would like to determine a sequence $x_1, x_2, \ldots$ of arguments such that $p = \sigma(x_1, x_2, \ldots)$. The general map $\sigma : (x_1, x_2, \ldots) \mapsto \sigma(x_1, x_2, \ldots)$ will be neither surjective nor injective. So synthesis is neither a synthesis of any imaginable p, nor is analysis unambiguous, i.e., the fiber of σ could be a large set of parameters. Moreover, one may have a synthesis operator σ_1 and an analysis operator σ_2 such that the analysis of a tame synthesis may become pathological.

In the following sections, we shall discuss four such operators: Fourier, frequency modulation, wavelets, and physical modeling. We shall however not deal with mixed synthesis/analysis problems which, mathematically speaking, are wild ones—let alone the associated technological problems.

A.1.2.1 Fourier

The Fourier approach deals with periodic functions, in our case periodic pressure variations $p(t)$ as functions of time t. This however is not the direct approach to produce a pressure function which is inserted into the support operator, since such a p has an infinite support. In order to turn a periodic function f into one with a standard support $Support(1, 0, 1)$, say, it is usually multiplied by an *envelope* function H, see Figure A.2. This is a continuous, piecewise differentiable[8] non-negative function on $\mathbb{R}$ which fits in the standard support

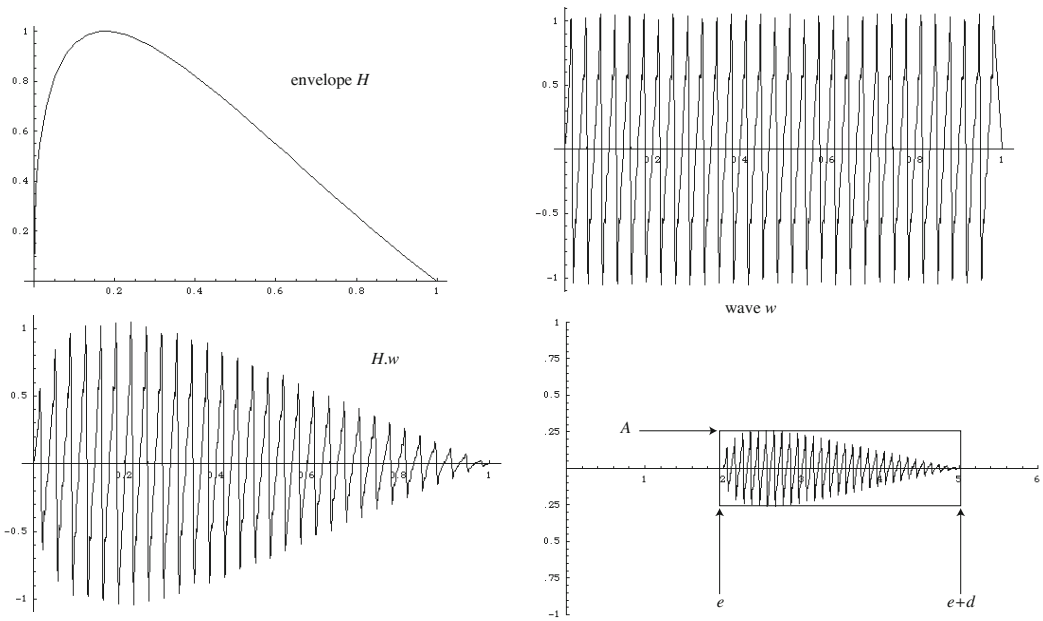

Fig. A.2. The envelope H (top left), the wave w (top right), and its combination $H.w$ (bottom left), as well as the affine deformation by the support operator $Support(A, e, d)$ (bottom right).

$Support(1, 0, 1)$, i.e., $H(t) = 0$ outside $[0, 1]$ and $\|H\|_\infty = Max(H) = 1$. In most technological applications, H is even a spline function (often even a linear, i.e., polygonal spline) modeling the attack and decay of a sound event. Then, the pressure function is given by $p = w.H$. Observe that this is already a source of

[8] Differentiable except for a finite set of points

poietic ambiguities: neither w (not even its frequency), nor H are uniquely determined by p. We then say that the standardized pressure function p is defined by the envelope H and the *wave* w.

Fourier's theorem deals with periodic wave functions w which are *piecewise smooth*[9] waves. For any piecewise smooth function $w : \mathbb{R} \to \mathbb{R}$, the additive group $\mathbb{R}$ acts by translations: $(e^P.w)(t) = w(t + P)$. The group of periods $Periods_w$ of w is the isotropy group of w under this action. For any non-zero period P, the inverse $f_P = 1/P$ is called a *frequency of* w, its unit is *Hertz*, Hz. If $P_w = inf(\{P \in Periods_w, 0 < P\}) = 0$, w is evidently constant, otherwise, $Periods_w = \langle P_w \rangle$ is the discrete group generated by the smallest positive period P_w. To avoid ambiguities in the periods or frequencies of a wave, one addresses this smallest period P_w or frequency $f_w = 1/P_w$ if one speaks about the "fundamental period" or the "fundamental frequency" of w (otherwise, not even the period or frequency of a wave would be uniquely determined). Fourier's theorem is this (for a proof, see [551]):

Theorem 45 *If w is a piecewise smooth wave function and $P \in Periods_w$ is a positive period, with $f = 1/P$ the corresponding frequency, then there are two sequences $(A_n, Ph_{n+1})_{n=0,1,2,\ldots}$ of real numbers such that*

$$w(t) = A_o + \sum_{n=1,2,3,\ldots} A_n \sin(2\pi n f t + Ph_n), \tag{A.6}$$

i.e., the infinite series converges and represents the wave for every time t for all points where the function is continuous. For the given period, the coefficients A_n and Ph_n are uniquely determined and can be calculated as follows:

$$\begin{aligned}
A_0 &= f \int_{-P/2}^{P/2} w(t)dt, \\
a_n &= 2f \int_{-P/2}^{P/2} w(t)\cos(2\pi n f t)dt, 0 < n, \\
b_n &= 2f \int_{-P/2}^{P/2} w(t)\sin(2\pi n f t)dt, 0 < n, \\
A_n &= \sqrt{a_n^2 + b_n^2}, 0 < n, \\
Ph_n &= \arcsin(a_n/A_n) \text{ if } A_n \neq 0 \text{ and } Ph_n = 0 \text{ else.}
\end{aligned}$$

The A_n is called the nth amplitude*, whereas Ph_n is called the nth* phase *of the wave with respect to the selected period. The sequence $(A_n)_n$ is called the* amplitude spectrum*, $(A_n^2)_n$ is called the* energy spectrum *since the energy of a wave is proportional to the square of the amplitude, and the sequence $(Ph_n)_n$ is called the* phase spectrum*. If the period/frequency is the fundamental period/frequency, one omits these specifications.*

An equivalent representation (with coefficients a_n, b_n, unique for a given period) is obtained for the explication of the sinoidal components via the goniometric formula

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

and yields

$$w(t) = A_o + \sum_{n=1,2,3,\ldots} a_n \cos(2\pi n f t) + b_n \sin(2\pi n f t). \tag{A.7}$$

Remark 31 It is well known *[617, Thm. 6.7.2]*, that the function sequences

$$(\sin(2\pi n f t))_{n=1,2,3,\ldots}, (\cos(2\pi n f t))_{n=0,1,2,\ldots}$$

[9] Continuous, except for a finite set of discontinuities (it need not be defined in these points, but the left and right limits of the functions exist in these points), with a continuous derivative, except for a finite set of points (where the derivative is not continuous or even not defined, but the left and right limits exist in all these singularities). Many examples are plain C^1 functions, but the saw-tooth function is not.

form an orthogonal basis of the pre-Hilbert space[10] $C^0[-P/2, P/2]$ of the continuous functions on $[-P/2, P/2]$ for the 2-norm (see Appendix I.1.2), where $f = 1/P$. This follows in particular from the trigonometric orthogonality relations of the defining scalar product $(f, g) = \int_{-P/2}^{P/2} f(t)g(t)dt$ of functions $f, g \in C^0[-P/2, P/2]$, i.e.,

$$(\sin(2\pi nft), \cos(2\pi mft)) = (\sin(2\pi nft), \sin(2\pi nft)) = (\cos(2\pi nft), \cos(2\pi mft)) = 0 \qquad \text{(A.8)}$$

for $n \neq m$. There is an infinity of such orthogonal bases for $C^0[-P/2, P/2]$, and mathematically, nothing distinguishes the sinoidal basis chosen by Fourier from the other orthogonal bases. Moreover, sinoidal functions are all but elementary. Mathematically, they are very complex, as is evident from Euler's identity $\cos(x) + i.\sin(x) = e^{ix}$. A justification for using sinoidal waves lies in the fact that simple mechanical differential equations, such as the spring equation $m.\ddot{x} = -k.x$, have sinoidal functions as their solutions. But this is a physical argument which must be coupled with a dynamical system of this equational type in order to give these functions any preference.

In order to meet the requirement for a unit amplitude, the coefficients of the Fourier representation can be dilated by a common factor, and we are done with the periodic wave. A common generalization of the Fourier representation (A.7) is defined if the frequency and coefficients are also functions of time: $f = f(t), A_n = A_n(t), Ph_n = Ph_n(t)$, a situation which is also needed to represent sounds of physical instruments with glissandi, crescendi, and their natural damping effects.

This construction is the poietic perspective. The esthesic one deals with the problem of constructing an envelope H, a periodic wave w and its Fourier representation (A.7) for a given sound function p. As was already mentioned above, the wave and the envelope cannot be reconstructed unambiguously in general. Even if the wave is known, the envelope is not reconstructible, although a number of obvious candidates can be calculated, e.g., a polygonal envelope defined by the local maxima and minima of the enveloped wave. As to the wave, one candidate for such can be guessed by the analysis of a time window $[t_1, t_2]$ of p within the supporting duration, such that the local maxima of p are relatively constant (neither at the initial, nor at the decay phase of the sound). One can then take a multiple of the period as a time window, and calculate the Fourier representation of this time window which is interpreted as a finite interval of a really periodic function, i.e., prolongation of this window to infinity. Although this period will not be the fundamental period of the wave, the Fourier representation will yield the right coefficients modulo a multiple of the fundamental frequency. If the fundamental period is small relatively to the total duration of the sound, there is a chance to calculate the underlying wave. In general, this is a highly ambiguous situation.

Once the wave is reconstructed, the Fourier coefficients are uniquely determined by the Fourier theorem, and we are done. But this is only the last phase of a highly ambiguous situation. It is of course always possible to find an underlying wave, it suffices to take the total duration of the support and to set it to the wave's period, i.e., prolongation to infinity of the sound by adding copies of itself to the left and to the right of the sound support. For the relation of these reconstructions to what is heard, see below Appendix B.

Remark 32 A final remark on the terminology of sound frequencies in music. Usually, when we deal with "the frequency of a sound", we do not mean that this frequency is a neutral property of the sound (although it could happen that the sound really has a fundamental frequency), but the fundamental frequency of a wave that is used in the standard representation of the sound as a product of its (periodic) wave and a deformed envelope. *This is a poietic definition, and this is what we will use because a neutral definition does not exist for general sounds.* In this setup, we have the sound function $p = Support(A, e, d)(H)w_{A.,Ph.}(f)$. Here, we may assume that $w_{A.,Ph.}$ is the formal representation of the trigonometric sum by the amplitude and phase spectra, and such that the total amplitude maximum is 1; the frequency is given as an additional argument, and the envelope H is given in its standard support. We call the 4-vector (e, f, d, A) the *geometric coordinates* of the sound, and the pair $(w_{A.,Ph.}, H)$ the *color coordinates* since they are responsible for the sound color (timbre) in this representation. For other poieses of sound which may also use the frequency coordinate (for example those to be discussed in the following sections), this one would also be referred to, but there is no neutral access to this concept.

[10] A normed real vector space whose norm is defined by a positive definite symmetric bilinear form.

A word of caution: The way humans "detect" sound frequencies is not neutral, this is an esthesic psychophysiological system whose function is far from understood, so do never mix up neutral facts with poietic or esthesic facts when dealing with sound attributes!

A.1.2.2 Frequency Modulation

In this section, we have a similar decomposition as discussed before: the sound p is written as a product $p = H.w$, where H is the envelope, and w is a "wave" function. However, this time we do not use the additive combination of sinoidal functions of Fourier synthesis to build w. The combination is rather a functional concatenation of such functions, i.e., sinoidal functions have in their arguments other sinoidal functions, and so on. This synthesis operator was introduced by John Chowning [198] and implemented first in the legendary Yamaha's DX7 synthesizers. The formal definition of frequency modulation (FM) functions in terms of circular denotators is given in Section 6.7, example 3, and yields this type of expressions:

$$FMsound(myFMObject)(t) = \sum_{i}^{n} A_i \sin(2\pi F_i t + Ph_i + FMsound(myModulator_i)(t))$$

where $myModulator_i$ is the FM-Object factor of the limit type denotator $Knot_i$. In the terminology of FM synthesis, the (respective) interior functions $FMsound(myModulator_i)(t)$ are called the *modulators* with respect to the (respective) exterior sinoidal functions $A_i \sin(2\pi F_i t + Ph_i)$ which are named the *carriers*. We symbolize this relation by an arrow $myModulator_i \Rightarrow myFMObject$. So the FM functions start as a sum of carriers where modulators are inserted, and these modulators are again of this nature, etc., until no modulator appears and the recursion terminates. In FM synthesis it is also allowed to have circular denotators in the sense that a modulator can be a carrier in one and the same function! Observe that the existence of a denotator $myFMObject$ describing such a "self-modulating" function does not imply the existence of the corresponding function $FMsound(myFMObject)$. However in digital sound synthesis, one often resolves this problem by taking the time argument of the modulator one digital step before the time argument of the carrier, i.e.,

$$FMsound(myFMObject)(t_n) = \sum_{i}^{n} A_i \sin(2\pi F_i t_n + Ph_i + FMsound(myModulator_i)(t_{n-1})).$$

In the DX7 implementation, the FM recursion scheme is presented in the graphical block diagram format of a so-called *algorithm* (not an adequate wording, though); Figure A.3 shows Yamaha's 32 algorithms.

Each sinoidal carrier component is written as a block, whereas the modulators of a given carrier are those blocks which are above the carrier and are connected by a line with this carrier. Each of the building blocks can be specified in the respective parameters.

The point of the FM synthesis is that it needs a small number of sinoidal functions—in fact only six sinoidal oscillators are needed for the DX7 algorithms—to simulate complex instrumental sounds, which is a theoretical and technological advantage.

But the elegant FM synthesis has also drawbacks concerning the uniqueness question: How many different denotators $myFMObject$ do yield the same function $FMsound(myFMObject)$? A general answer seems difficult, we have two partial solutions. The first regards a tower of modulators:

Lemma 65 *Let $G : \mathbb{R} \to \mathbb{R}$ be a bounded C^1 function. Then G is determined by its value $G(0)$ and by any function $F : \mathbb{R} \to \mathbb{R}$ of the shape $F(t) = a + b\sin(ct + G(t))$, with $b \neq 0$.*

Proof. Evidently, by the boundedness of G, $a+b = max(F), a-b = min(F)$, so $a = (max(F)+min(F))/2, b = (max(F) - min(F))/2$, are determined by F. Moreover, the continuous function $G' = \frac{bF'}{\sqrt{b^2-(F-a)^2}} - c$ is a function of F, and the value $G(0)$, together with the integral of G' which is a function of F, completely determine G, QED.

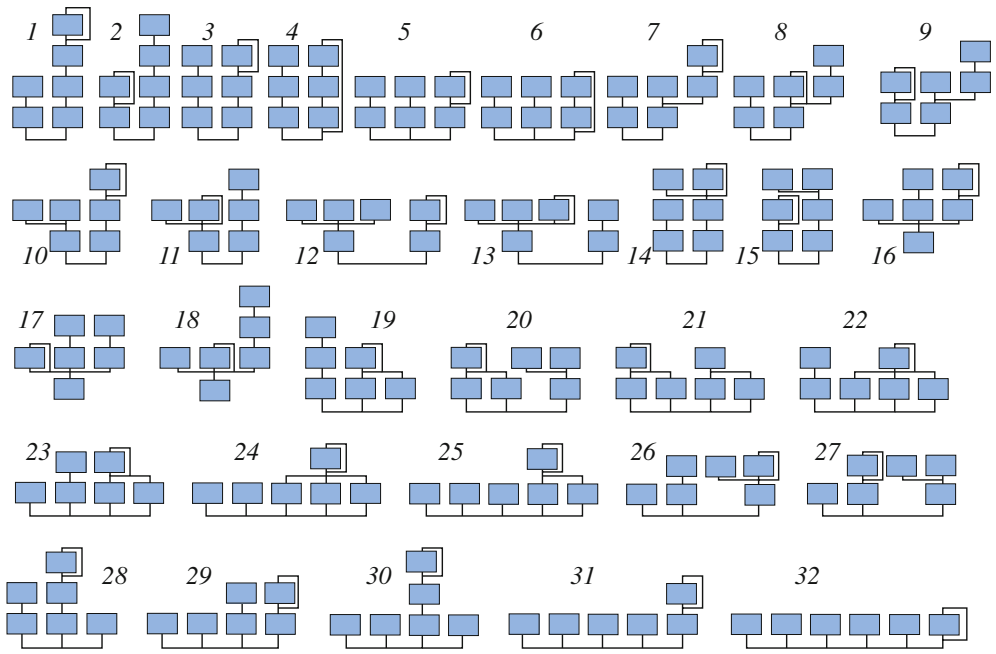

Fig. A.3. Yamaha's 32 algorithms for FM synthesis. Only six sinoidal oscillators are required to generate a variety of more or less natural sounds.

Proposition 73 *If $M = (M_n \Rightarrow M_{n-1} \Rightarrow M_{n-2} \Rightarrow \ldots M_0)$ is an FM denotator defined by a sequence of relative modulators for the functions $M_i = A_i \sin(2\pi f_i t + ?)$, then the resulting sound function $FMsound(M)$ determines all the modulators.*

Proof. This follows from lemma 65 by recursion on modulators, starting at total function $FMsound(M)(t) = A_0 \sin(2\pi f_0 t + FMsound(M_1))$, where M_1 is the denotator starting from M_1 instead of M_0. In the lemma, take $F = FMsound(M), G = FMsound(M_1)$, and observe that $G(0) = 0$, QED.

The second solution regards a flat sequence of unrelated carriers:

Proposition 74 *Let $F(t) = \sum_{i=1,\ldots k} a_i \sin(b_i t)$ with $0 < b_1 < \ldots b_k, 0 < a_i$ be the function associated with a flat FM denotator. Then the coefficients, and therefore the denotator, are all uniquely determined by F.*

Proof. We have $(-1)^n F^{2n+1}(0) = \sum_{i=1,\ldots k} a_i b_i^{2n+1}$ for all $n \geqq 1$. Consider the function $H(a.,b.)(x) = \sum_{i=1,\ldots k} a_i b_i^x$ of the real variable x. Suppose we have sequences $(a_i)_{i=1,\ldots k}, (b_i)_{i=1,\ldots k}$ and $(a'_i)_{i=1,\ldots k}, (b'_i)_{i=1,\ldots k}$ such that they yield the same function F. Then we have $H(a.,b.)(x) = H(a'.,b'.)(x)$ for all $x = 2n+1, n \geqq 1$. Write $b_i = e^{\beta_i}, b'_i = e^{\beta'_i}$, and suppose $b_k < b'_k$. Then

$$\frac{H(a.,b.)(x)}{H(a'.,b'.)(x)} = \frac{\sum_{i=1,\ldots k} a_i e^{\beta_i x}}{\sum_{i=1,\ldots k} a'_i e^{\beta'_i x}} = \frac{\sum_{i=1,\ldots k} a_i e^{(\beta_i - \beta_k)x}}{\sum_{i=1,\ldots k} a'_i e^{(\beta'_i - \beta_k)x}} = \frac{\sum_{i=1,\ldots k-1} a_i e^{(\beta_i - \beta_k)x} + a_k}{\sum_{i=1,\ldots k} a'_i e^{(\beta'_i - \beta_k)x}}.$$

But for $x \to \infty$, the denominator goes to 0 whereas the numerator goes to a_k, contradicting the fact that this quotient is 1 for all $x = 2n+1, n \geqq 1$. Therefore $b_k = b'_k$, so in the above quotient, the limit for $x \to \infty$ is a_k / a'_k which is also 1, and we have equal coefficients for the index k. Therefore, we may proceed by induction to $k-1$ and we are done, QED.

These very special results show that given the tower or the flat FM schemes, the functions determine their coefficients (the background denotators) uniquely, but for general FM schemes, there is no such a result. Moreover, if the FM scheme is not known, we have no idea of how the scheme should be determined from the

function. This is a drawback compared to the Fourier operator, where the coefficients are always uniquely determined once the fundamental frequency is fixed. In other words, FM synthesis is much more efficient than Fourier synthesis, but one has to pay for this when turning to the respective analyses.

The switch between Fourier and FM operators is essentially managed by Bessel functions. These are defined directly from a core situation from FM synthesis, i.e.,

Definition 125 *Let z be a real number. Then the Fourier expansion of the 2π-periodic function $\sin(z\sin(t))$ of t is*

$$\sin(z\sin(t)) = 2\sum_{n=0,1,2,\ldots} J_{2n+1}(z)sin((2n+1)t), \tag{A.9}$$

whereas the Fourier expansion of the 2π-periodic function $\cos(z\sin(t))$ of t is

$$\cos(z\sin(t)) = J_0(z) + 2\sum_{n=1,2,3,\ldots} J_{2n}(z)cos(2nt). \tag{A.10}$$

The functions $J_m, m = 0, 1, 2, \ldots$ are called the mth Bessel functions.

The above definition relies on the Fourier representation of the respective functions and on their properties as odd or even functions. An alternative definition of Bessel functions is $J_m(z) = \frac{1}{\pi}\int_0^\pi \cos(mt - z\sin(t))dt$.

From Definition 125, one obtains the following fundamental equations for Bessel functions:

$$\sin(r + z\sin(s)) = \sum_{-\infty}^{\infty} J_n(z)\sin(r + ns),$$

$$\cos(r + z\sin(s)) = \sum_{-\infty}^{\infty} J_n(z)\cos(r + ns).$$

In particular, with $s = 2\pi ft, z = I, t = 2\pi gt$, we have

$$\sin(2\pi ft + I\sin(2\pi gt)) = \sum_{-\infty}^{\infty} J_n(I)\sin(2\pi(f + n.2\pi g)t),$$

which is a Fourier type linearization, however, it is not a proper Fourier representation since the so-called "nth side band" frequencies $f + n.2\pi g$ are not a multiple of a fundamental frequency in general. This is rather a reduction of a FM concatenation to a flat FM configuration. Conversely, every (finite) Fourier decomposition is evidently a flat FM configuration.

A.1.2.3 Wavelets

Although the FM operator is much more efficient than the Fourier operator, it is still an operator which produces functions with an infinite support, a property which no real sound shares, and we have in fact added an envelope to cope with this requirement for Fourier and FM operators. From this point of view, wavelets are fundamentally better suited for handling finite sound objects without any envelope casting. Refer to [622] and [559] for the wavelet theory and its applications to sound and music. Let f be a square integrable function (element of $L^2(\mathbb{R})$, i.e., $\int_{\mathbb{R}} |f(x)|^2 dx < \infty$). Then its Fourier transform is defined and is the function

$$\hat{f}(\omega) = (2\pi)^{-1}/2\int_{\mathbb{R}} f(x)e^{-ix\omega}dx. \tag{A.11}$$

Definition 126 *A square integrable function ψ is called a* wavelet *if $0 < c_\psi = \int_{\mathbb{R}} \frac{|\hat{\psi}|^2}{|\psi|}d\psi < \infty$. For a wavelet ψ, the* wavelet-transformed *of a square integrable function f is the function (of two variables a, b)*

$$L_\psi f(a, b) = c_\psi^{-1/2}|a|^{-1/2}\int_{\mathbb{R}} f(t)\psi(\frac{t-b}{a})dt, \tag{A.12}$$

with $a \in \mathbb{R} - \{0\}, b \in \mathbb{R}$.

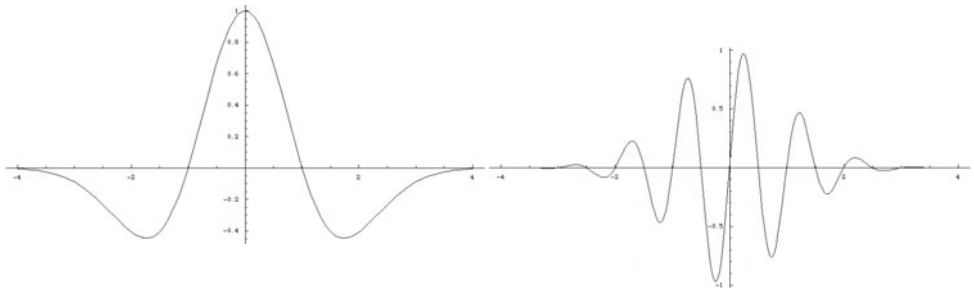

Fig. A.4. Two wavelets: Murenzi's Mexican hat [776] (left) and Morlet's wavelet (imaginary part) [385] deduced from the sinoidal function (right).

Figure A.4 shows two typical examples of wavelets. The wavelet transform is a function of two variables defined on every couple $(a, b) \in (\mathbb{R} - \{0\}) \times \mathbb{R}$. The point of this representation is that it is a kind of system of coefficients $(L_\psi f(a, b))_{(a,b)\in\mathbb{R}-\{0\}\times\mathbb{R}}$ which is parametrized by two real numbers, corresponding to a scalar product

$$c_\psi^{-1/2}(f, Support(|a|^{-1/2}, b, a)(\psi)) \tag{A.13}$$

of the function f with an affinely deformed version[11] $Support(|a|^{-1/2}, b, a)(\psi)$ of the "mother" wavelet ψ. This deformation $Support(|a|^{-1/2}, b, a)$ is an isometry on the square integrable functions, see also Figure A.5 for some deformed wavelets. By the following formula, the wavelet transformations $L_\psi f(a, b)$ redetermine

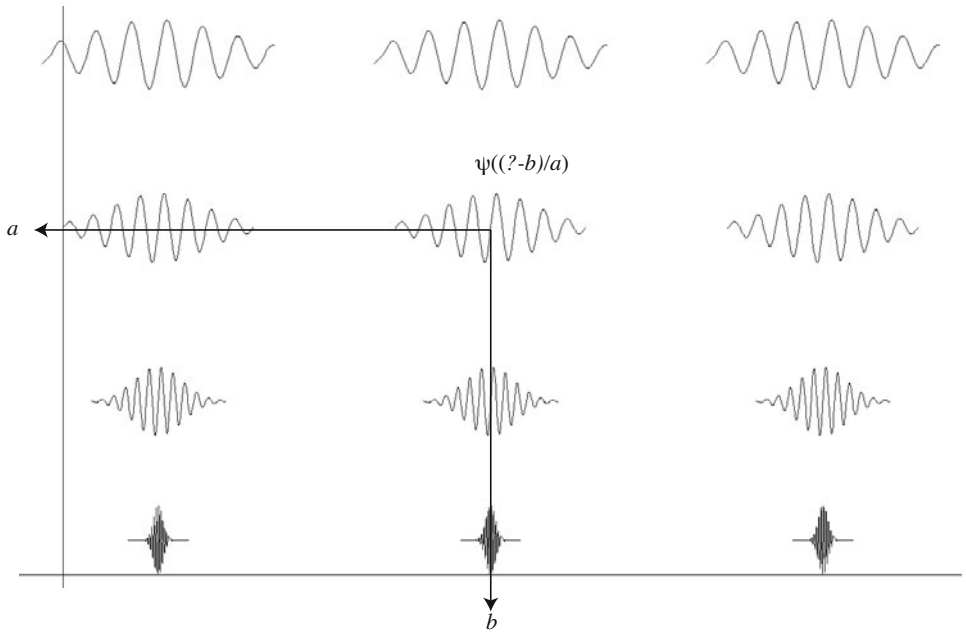

Fig. A.5. Various affine deformations of the mother wavelet ψ.

the original function f:

[11] Observe that the operator $Support(|a|^{-1/2}, b, a)$ defines a linear action $Support : \overrightarrow{GL}(\mathbb{R}) \to GL(L^2(\mathbb{R})) : e^b a \mapsto Support(|a|^{-1/2}, b, a)$ of the affine group $\overrightarrow{GL}(\mathbb{R})$ on the space $L^2(\mathbb{R})$ of square integrable functions.

$$f(t) = c_{\psi}^{-1/2} \int_{\mathbb{R}^2} L_{\psi} f(a,b) Support(|a|^{-1/2}, b, a)(\psi)(t) \frac{da\ db}{a^2}. \tag{A.14}$$

For selected wavelets, it is possible to generate an orthonormal basis of $L^2(\mathbb{R})$ which is defined as a so-called *frame*. For $a_0 > 1, b_0 > 0$, and for the Meyer wavelet[12] ψ we consider the frame $(Support(a_0^{-m/2}, nb_0a_0^m, a_0^m)(\psi))_{m,n\in\mathbb{Z}}$ of deformed versions of ψ. Then this is an orthonormal basis of $L^2(\mathbb{R})$ for $a_0 = 2, b_0 = 1$. This is an analogous situation as encountered for Fourier series and their sinoidal bases with the known trigonometric orthogonality relations in formulas (A.8).

For a comparison of Fourier and wavelet analysis, see [1105].

A.1.2.4 Some Remarks on Physical Modeling

The previous operators were directly acting on the production of a sound function. Their poietic nature was a mathematical one, just to construct a time function $p(t)$ by a mathematical procedure from a certain type of "atomic", i.e., basis functions. In contrast, physical modeling is one step more poietic in that it does not directly deal with sound, but with a physical system that produces sound. On the one hand, this is a strong restriction since sound does not care for the physical system that evokes that sound. It seems however that this drawback is compensated by the fact that musical expressivity is largely determined by the physical device which the artist manipulates when interpreting or improvising music. It is also an important argument that the simulation of the physical instrument and then a possible canonical extension could yield more interesting sounds than just "abstract nonsense" procedures. For a general survey on physical modeling, see [911, Chapter 7], we restrict our discussion of this extensive topic to some systematic remarks.

The idea is that one considers a physical model of a sound production device and then implements this model as a software which—if sufficient calculation power is available—calculates the physical output on the level of the sound wave that is emitted by the modeled instrument. At present, there are three methodologies for such a modeling: mass-spring, modal synthesis, and waveguide. The mass-spring paradigm just models a physical instrument (a string, a drum) by a finite space configuration of point masses that are related by springs and damping effects [469]. The modeling is built upon the classical mechanics of Newton's law and the corresponding dynamic behavior that eventually terminates in the air's vibration.

The modal synthesis paradigm [171] reduces the vibrating physical system to a system of vibrating substructures, usually very small in number compared with the mass-spring components. These substructures are characterized by their frequencies, damping coefficients and parameters for the vibrating mode's shape. This adds up to a sum of modal vibrations. Whereas in simple configurations these data can be obtained from classical literature in equations for vibrating systems, the complex data must be extracted from experimental results. A prototypical implementation of this paradigm is MOSAIC, developed by Jean-Marie Adrien and Joseph Morrison, see [911, p.276 ff.].

The waveguide paradigm has been implemented in commercial physical modeling synthesizers by YAMAHA and KORG , see [911, p.282 ff.]. It has mainly been developed by Julius O Smith III and collaborators [989, 990]. The waveguide model implements the traveling wave along a medium, such as a tube or a string. See [991] for an update of physical modeling strategies.

Although physical modeling is a successful approach in the simulation of musical instrumental sounds, it is a step back from the neutral sound objects to their generators. This has not only been a technological requirement for performance theory (where physical modeling is a core approach), it is also a consequence of the failure in the understanding of the topological semantics of sound objects as exposed in Section 12.3. In particular, it is an open problem to relate the physical modeling theory and technology to the other operators, such as Fourier, FM, and wavelets. The deeper question here is whether the neutral sound objects are really the most relevant ingredients of musical performance, i.e., how strongly the gestural components influence and characterize the sounding reality. It is not clear how deeply a sound conveys the generating gestures in its autonomous structure. To our knowledge, sound classification has not been directed towards a gestural coordinate in the neutral sound description, except, perhaps, in the straightforward envelope component.

[12] See [622, 2.1.25].

A.2 Mathematical and Symbolic Spaces

The physical description of sounds is not what can be used for music theories. This is based on (1) the way humans perceive sounds, (2) the shape of music thinking, and (3) available instrumental technologies. Therefore physical parameter spaces must be transformed into spaces which essentially encode the same information, but do so in a way which is more adapted to music. Basically, we shall present mathematical structures where physical parameters are represented. Based upon these spaces, we shall explain a derived set of representations which encode different music-topographic aspects. Our discussion regards the geometric coordinate pairs (basis coordinate plus associated pianola coordinate) onset and duration (A.2.1), amplitude and crescendo (A.2.2), and frequency and glissando (A.2.3).

A.2.1 Onset and Duration

In music, one speaks about tempo, metronome, quavers, semiquavers, triplets, 3/4 meter, etc. The relation to the physical time parameters is as follows. To begin with, the *physical* onset time e *sec* and duration d *sec* are opposed to a *musical* onset time E *note* and duration D *note*, usually also in real values, and in units such as "*note*", meaning a whole note. However, in many practical contexts, the rational number field $\mathbb{Q}$ will do. In this latter context, we have the ratios of integer numbers $E = w/n, D = z/n, n > 0$, and the denominator is of the form $n = 2^r 3^s 5^t 7^u$ with natural exponents r, s, t, u, whereas the numerators w, z are integers, i.e., we are working in the ring localization[13] $\mathbb{Z}[1/2, 1/3, 1/5, 1/7]$ at the primes $2, 3, 5, 7$. Musically, this means that one is allowed to add, subtract, multiply such numbers at will without leaving the domain, and also division of such numbers by $2, 3, 5, 7$ leaves the domain invariant, i.e., the construction of duplets, triplets, quintuplets, septuplets is possible without any restriction.

The musical time shares a mental reality and should not be confused with physical time. Genetically, musical time is an abstraction from physical time, but they are by no means equivalent. This abstraction is also a creation of an autonomous time quality where mental constructions such as a score can be positioned. The relation between these two time qualities is defined by the tempo, usually encoded as a metronomic indication of x quarters per minute according to Mälzel, meaning a difference quotient (velocity) $\Delta E/\Delta e$ of musical time per physical time. This presupposes that we are given a one-to-one performance mapping $E \mapsto e(E)$ from musical time to physical time. *It is common in mathematical music theory to write physical parameters in lower case letters, whereas musical parameters are written in upper case letters.*

Let x, y be positive integers. Then every onset E can be written uniquely as $E = \delta + \tau.x/y$, where $0 \leqslant \delta < x/y$ is in $\mathbb{Q}$, and $\tau \in \mathbb{Z}$. If $\delta = q/(ny)$, we say that *in* x/y *time*[14]*, E is on the* $(q+1)$*st* ny*-tuplet in bar* $\tau + 1$. The additive group $\mathbb{Z}.1/y$ is called the *meter* of the x/y time. Evidently, this initializes a score at time $E = 0$, but this is pure convention. Pay attention *not* to view the symbol x/y as a plain fraction, but as a pair x, y giving rise to a fraction. In other words, in music notation, the symbol x/y is the mathematical fraction x/y *plus* the meter.

Example 91 Let $x = 3, y = 4, E = 15.375$. Then we have $E = 3/8 + 20.3/4$, i.e., E is on the fourth quaver of bar 21.

If we work in $\mathbb{Z}[1/2, 1/3, 1/5, 1/7]$, then with E and $1/y$, the remainder δ and the meter are automatically in this domain. If we are given a duration D, then durations of form D/n are also called n-tuplets (with respect to D).

The reason why the concept of onset time is not common in music lies in the fact that a note's onset can be deduced from its position in the score. The time signature, bar-lines, and simultaneous or preceding notes help establish the time context of each note. This helps calculate onsets algorithmically by recursion from the first score onset. This algorithm is derived from the linear syntagm of written language, but it sometimes leads to ambiguities.

[13] Make the integers $2, 3, 5, 7$ invertible, i.e., admitting fractions $1/2, 1/3, 1/5, 1/7$. But see Appendix E.4.1 for a formal definition.

[14] We stick to the continental terminology since "meter" will be reserved for a different concept.

A.2.2 Amplitude and Crescendo

The amplitude A of a sound relates to the loudness sensation, i.e., to musical dynamics. However, the chain of transformations to the physiologically relevant measures is quite complex [914]. The first member of this chain is the transformation which associates A with the *sound pressure level* (SPL)

$$l(A) = 20.\log_{10}(A/A_{threshold})\ dB. \tag{A.15}$$

Here, $A_{threshold} = 2.10^5\ Nm^{-2}$ is close to the amplitude of the threshold pressure variation for a 1 kHz sound in a young, normal hearing human subject. The unit dB for l is *Dezibel*. Of course, these constants are pure convention. From the auditory physiology, the shape $l(A) = a\ln(A) + b$ is essential, i.e., $l(A)$ is a conventionally normed linear function of the logarithm of the amplitude. The physio-psychological motivation for this approach lies in the Weber-Fechner law according to which the sensation of a difference of sensory stimuli is proportional to the stimuli [494], yielding the logarithmic representation as an adequate encoding of this sensorial modality. Figure A.6 shows some environmental SPL values (the musical units *ppp, pp*, etc. will be discussed below). For example, the SPL $\approx$ 120 dB of an air jet is one million times the threshold SPL.

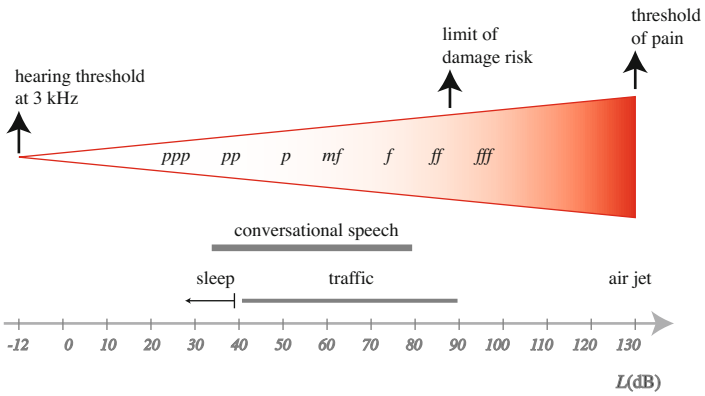

Fig. A.6. Some environmental sound pressure level values.

Just as duration is the time interval from the onset time to the "offset" time of a sound event, the *crescendo* parameter c is the difference between the onset SPL and the offset SPL. In normal piano sounds, this vanishes, but for violins, trombones, etc., this is a relevant quantity. In this context, one supposes that the sound representation $p = Support(A, e, d)(w_{A.,Ph.}(f).H)$ has a time-dependent amplitude $A = A(t)$. At the offset time $e + d$, the amplitude has changed by the amount $A(e + d)/A(e)$, or in terms of loudness, $c = l(A(e + d)) - l(A(e))$. As with onset and duration, it is not specified what happens in the sound process between the onset and the offset, it's just the difference that matters. This difference c is termed the (physical) *crescendo*.

In the musical abstraction L for loudness, one also works in the field $\mathbb{R}$ of real numbers, but the common score notation is not well defined for all real values. More precisely, the first still physically motivated codification is a conventional calibration such as, for example, $l : \mathbb{Z} \to \mathbb{R}$ with $l(L) = 10\ dB.L + 60\ dB$ such that the values

$$\begin{array}{ll} & 0 \mapsto mf \\ 1 \mapsto f & -1 \mapsto p \\ 2 \mapsto ff & -2 \mapsto pp \\ 3 \mapsto fff & -3 \mapsto ppp \end{array}$$

are arranged symmetrically around the *mezzoforte* sign. This looks like an identification of the score symbols with precise physical values. But this is wrong, it is a transformation from mental to physical reality. In fact, the affine transformation $l(L)$ is only a "default" assignment of a mental quantity symbolized by a dynamic symbol (which is codified by an integer). This fact is more evident if one recalls the velocity parameter in the MIDI code. This quantity is an integer in the interval $[0, 127]$, but the physical meaning of a velocity value depends on the assignment by specific technological calibrations, in particular by the output chain where the loudspeakers can take values that have nothing to do with the velocities. Units in the mental level of loudness are not standard, but we could, for example, take *vel* for MIDI's velocity parameter.

The extension of integer values to real or rational values (depending on the specific usage) is a reasonable procedure, for instance preconized by MIDI velocity codification. It is also useful for finer loudness data management with relative loudness signs such as "crescendo", "diminuendo" which make sense in a mental crescendo parameter which we shall denote by C. See also our discussion of performance transformations and primavista weights in Section 39.2.

A.2.3 Frequency and Glissando

Sounds which share a frequency parameter f are musically important, at least in the European tradition. However, the corresponding mathematics is a bit more involved than for time and amplitude. This is an expression of an intense discussion of traditional harmonies and interval theories with an evolving instrumental technology.

To begin with, f behaves like amplitude: For reasons of auditory physiology (see also Appendix B), f is transformed via the logarithmic formula $h(f) = u\ln(f) + v$, yielding the *physical pitch* h of frequency f. The common unit of pitch is the *Cent* Ct, it corresponds to the logarithm of a relative frequency increase by the factor $2^{1/1200}$, i.e., one percent of a well-tempered semitone (see below for tuning types); this entails $u = 1200/\ln(2)$, but the constants u, v are purely conventional. Presently, and in the Western framework, the relevant frequencies have the shape $f = 132\ Hz.2^p.3^s.5^t$, where $p, s, r \in \mathbb{Q}$. It is based on the *chamber pitch* $440\ Hz$ of the one-line a, as fixed in London 1939. Instead, we have chosen the (unlined) c with the frequency $132\ Hz = 440\ Hz.2^{-1}.3.5^{-1}$ as a starting frequency in order to relate the examples with ease to c. Mathematically, the restriction to the first three primes $2, 3, 5$ is not essential. One could as well take any sequence $p_1, p_2, \ldots$ of mutually prime natural numbers (larger than 1), the rational powers of which are multiplied by $132\ Hz$.

The natural logarithm of f is

$$\ln(f) = p.\ln(2) + s.\ln(3) + r.\ln(5) + \ln(132).$$

But music theory is rather interested in the relative pitch, i.e.,

$$\ln(f) - \ln(132) = p.\ln(2) + s.\ln(3) + r.\ln(5).$$

Moreover, the passage to another logarithmic basis b could be desirable, i.e.,

$$\ln(b)^{-1}.\ln(f) - \log_b(132) = p.\log_b(2) + s.\log_b(3) + r.\log_b(5),$$

so that we are given the linear function

$$u.X + v = \ln(b)^{-1}.X - \log_b(132)$$

for $X = ln(f)$. For music theory, the restriction to rational exponents is essential, since this hypothesis enables an unambiguous representation, see Appendix E.2.1. Every frequency f can be replaced by a point $x = (p, s, r) \in \mathbb{Q}^3$. Such a point represents the frequency $f(x) = 132\ Hz.2^p.3^s.5^r$, and from $f(x) = f(y)$, we conclude $x = y$. With this interpretation, $\mathbb{Q}^3$ is called the *Euler space* and a point $x = (p, s, r) \in \mathbb{Q}^3$ is called an *Euler point*. This means that a real number $p.\log_b(2) + s.\log_b(3) + r.\log_b(5)$ is viewed as a vector which is a rational linear combination of linearly independent vectors $\log_b(2), \log_b(3), \log_b(5)$.

The choice of the three first primes stems from the tradition of just tuning, where for two frequencies f, g, we have:

$$(O) f/g = 2/1 : f \text{ is the } \textit{octave} \text{ (frequency) for } g.$$
$$(Q) f/g = 3/2 : f \text{ is the (just) } \textit{fifth} \text{ (frequency) for } g.$$
$$(T) f/g = 5/4 : f \text{ is the (just) } \textit{major third} \text{ (frequency) for } g.$$

This is why 2 is associated with the octave, 3 with the fifth, and 5 with the major third. We therefore call p the *octave coordinate*, r the *fifth coordinate*, and s the *third coordinate* (of the sound with frequency f or of the corresponding Euler point. The Euler point $o = (1, 0, 0)$ is called the *octave point*, the point $q = (0, 1, 0)$ is called the *fifth point*, and $t = (0, 0, 1)$ is called the *third point*. Observe that fifth and third points are not the fifth and octave. The structural meaning of these points is explained in Section 6.4.1. The minor just third frequency 6/5 does not add a new prime number.

For a fixed chamber pitch, the *2-3-5 just tuning* (short: just tuning) is the set of frequencies which are represented by integer coordinates, i.e., the subgroup $\mathbb{Z}^3 \subset \mathbb{Q}^3$ of the Euler space. This is the three-dimensional grid which was introduced by Leonhard Euler [307]. The group of those grid points whose third coordinate vanishes is the *Pythagorean tuning*.

The Euler space is derived from the logarithmic representation of pitch, but the coefficients are beyond physical reality, this is why we view this space as a mental space. The same is valid if we consider the codomain space of pitch h, but interpreted with rational coefficients (written as $\mathbb{R}_{[\mathbb{Q}]}$ instead of the usual real line $\mathbb{R}$, the one-dimensional real vector space[15]). More precisely, one should name H the pitch on the mental pitch space $Pitch$, and h the pitch on the physical pitch space $PhysPitch$. Whereas the unit for physical pitch is $Cent$, the unit for mental pitch could be $Semitone$ or MIDI's key(number), but no standard exists here. In performance, we have a transformation $\wp : Pitch \to PhysPitch$. Both spaces have the real numbers as underlying sets, but the meaning of the spaces is different. The mental pitch space encodes pitch as it is symbolized on the score or as a key number in MIDI code. Ideally, performance transforms this abstract data into mathematical pitch $Math(H) = (p, s, r)$ in the Euler space, and this is transformed to the pitch $\ln(f) = p.\ln(2) + s.\ln(3) + r.\ln(5) + \ln(132)$ in $\mathbb{R}_{[\mathbb{Q}]}$. Only after forgetting about the coefficients, we are in $PhysPitch$. And this is by no means a formal play: It is a dramatic change of reality if one views the reals as an infinite-dimensional space with linearly independent octave, fifth, and third logarithm vectors, or as a line, where everything shares the same direction! In analogy to onset and loudness, one introduces physical glissando g and its symbolic counterpart G.

In the common visualization of the Euler space, all grid cells are shown as cubes. This could suggest that angles and distances are relevant to this space. So far, this has however no musicological reason, and it is nonsense to argue with angles and distances as Arthur von Oettingen [806], Carl Eitz[297], and Martin Vogel [1089] have done.

Evidently, the mathematical structure of just tuning is independent of the historical choice of the sequence 2,3,5. One could as well take any pairwise prime positive numbers $1 < p_1 < p_2 < p_3$ and would get a p_1-p_2-p_3 just tuning.

If the pitch range in Euler space is described by non-integer coefficients, one speaks of *tempered tunings*. The most well known are defined by a uniform construction mode. For a natural number $w > 1$ one considers all pitches whose octave coordinate is a fraction of shape $p = x/w$, whereas the other coordinates vanish. This is called the *w-tempered tuning*. This defines a grid $\mathbb{Z}.1/w.o \xrightarrow{\sim} \mathbb{Z}$ with step width $1/w$ of the octave point. By the same recipe, tempered tunings in fifth and third direction can be defined. The 12-tempered tuning is the famous "well-tempered" tuning. The 1200-tempered tuning is less interesting for conventional composers than for measurement techniques, where the unit step is the Ct step defined above.

More generally, one may define tunings which consist of tempered and just components. The procedure runs as follows. Take three positive integers w_1, w_2, w_3 and consider the grid

$$\mathbb{Z}.\frac{1}{w_1}.o + \mathbb{Z}.\frac{1}{w_2}.q + \mathbb{Z}.\frac{1}{w_3}.t$$

[15] See Appendix E.2.1.

which specializes to the 2-3-5 just tuning as well as the tempered tunings as defined above. Call this construction the w_1-w_2-w_3 *just-tempered tuning*. Historically relevant is the *mediante* tuning, which is the 1-1-2 just-tempered tuning, and which includes the tempered whole-tone step in the major third.

With respect to the auditory psycho-physiology (see Appendix B), we should consider the distribution of the just tuning grid vectors $x = (p, s, r)$ in $\mathbb{Z}^3$ with respect to the mathematical pitch $H(x) = p.\log_b(2) + s.\log_b(3) + r.\log_b(5)$. More generally, take any vector $x \in \mathbb{R}^3$ and denote by $H_{prime} = (\log_b(2), \log_b(3), \log_b(5))$ the *prime vector*. This means that pitch is the usual scalar product

$$H(x) = (H_{prime}, x) \tag{A.16}$$

of the prime vector with the generalized Euler point x. Therefore,

Proposition 75 *With the definition* (A.16) *two vectors* $x, x' \in \mathbb{R}^3$ *have the same pitch* $H(x) = H(x')$ *iff their difference is orthogonal to the prime vector* H_{prime}.

Call $E = H_{prime}^{\perp}$ the plane orthogonal to the prime vector. Then the proposition means that for any generalized Euler point x, $x + E$ is the set of points with same pitch as x. Now, according to what we know, E lies so skew in $\mathbb{R}^3$ that for a point $x \in \mathbb{Q}^3$,

$$(x + E) \cap \mathbb{Q}^3 = \{x\}.$$

Nonetheless, every real number ϕ can be approached within any given error by points of the just tuning grid: For any positive bound δ, there is $x \in \mathbb{Z}^3$ such that $|H(x) - \phi| < \delta$, see Appendix D.4 for a proof. In particular, we have the following proposition which has dramatic consequences for theories of hearing of just tuned pitch (see Section B.2):

Proposition 76 *If* $\phi = H(x_0)$ *is a mathematical pitch of a grid point* x_0*, then for any positive bound* δ*, there is an infinity of grid points* x *such that* $|H(x) - H(x_0)| < \delta$.

B

Auditory Physiology and Psychology

"Music listening" is a metonymy of understanding music: For all participants, the ear functions as an interface for perceiving music between physical, psychological, and mathematical reality. But a metonymy is not the matter as such. This is what deaf Beethoven teaches us impressively: his innermost ear was an organ of imagination that was uncoupled from the material ear. And the physiology of the hearing process teaches us that the neural coupling of the ear to the respective cortical regions is extremely complex and still hardly understood. Section B.1 is written to give an overview on auditory physiology.

Beyond receptive processes hearing means also an active shaping according to templates of esthesis and poiesis. One of the difficult basic problems concerning the activity of hearing deals with the compatibility of these templates and the physical input, and in particular the notorious "straightened out hearing" (German: "Zurechthören") which keeps alive a lot of wishful thinking in music theory. Basically, the problem is that we do not perceive physical sounds, but classes of indistinguishable sounds, what Werner Meyer-Eppler coined "valences" in [735]. Valence theory, which is sketched in Section B.2, has dramatic consequences for the relation between mathematical theories and their semantic potential for music.

Historically and materially, and in view of the genealogy of a mathematical theory, the subject pairing of "consonance-dissonance" is an excellent illustration of auditory physiology and psychology. We deal with the formal and some physiological aspects of this subject in part VII. In the following Section B.3, we want to expose the stratification of the phenomenon of consonance and dissonance. Hereby, the problem setup as well as the approaches to its solution demonstrate a strong dependency of the addressed reality layer. Methodologically, Section B.3 is important since it makes evident that known approaches of the mathematical argumentation in musicology turn out to be too narrow with respect to the existing music, and too dogmatic and scientifically unbased with respect to music thinking.

B.1 Physiology: From the Auricle to Heschl's Gyri

Two fundamentally different regions of sound processing in the auditory system can be distinguished. In the peripheral region, mechanical preprocessing takes place, especially in the fluid-filled inner ear. The sensory cells encode the preprocessed mechanical oscillations into electrical nerve action potentials. In the second region of the hearing system, neural processing of the sound information is conducted in ascending nuclei, which finally leads to auditory sensation. The hearing system, especially in mammals, has pushed its bandwidth up in a frequency range, where the limits of neural processing in the time-domain are exceeded by far. Instead of processing high frequencies in the time domain, evolution has developed the so-called frequency-place principle. In the inner ear, the frequency contents of sound signals are separated in the spatial domain. The sensitivity of our hearing system is thereby remarkable, in its most sensitive region, the threshold of the hearing system is limited only by thermal noise.

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B.1.1 Outer Ear

Sound energy is collected by the outer ear and transmitted through the outer ear canal to the ear drum (Figure B.1).

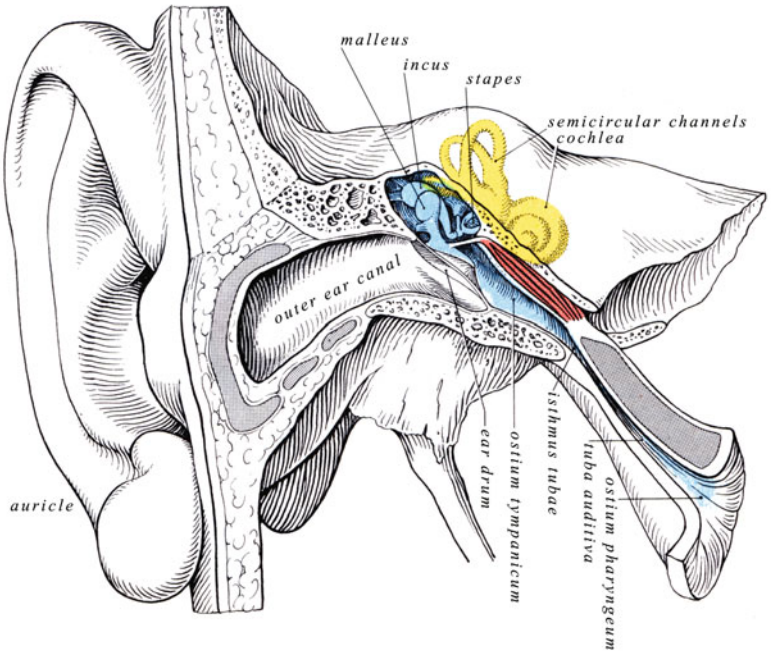

Fig. B.1. Schematic view of the outer, middle and inner ear, modified from [1157], © Springer-Verlag Berlin Heidelberg 1999. With permission of Springer.

The delicate structures of our hearing system are well protected inside the skull. For the sound transmission, the outer ear canal acts like an open pipe with a length of about 20 to 30 mm. Its quarter-wave resonance is responsible for the high sensitivity of our hearing organ in this frequency range, indicated by the dip of the threshold in quiet around 4 kHz. This high sensitivity is however also the reason for high susceptibility to noise-induced damage in the region around 4 kHz.

B.1.2 Middle Ear

The fluids of the inner ear must be excited by the sound-induced vibrations of the air particles in front of the ear drum. The light but sturdy funnel-shaped ear drum (tympanic membrane) operates over a wide frequency range as a pressure receiver. It is firmly attached to the long arm of the hammer (malleus) (Figure B.1). The motions of the eardrum are so transmitted via the anvil (incus) to the stirrup (stapes). The stapes foot plate, together with a ring-shaped membrane called the oval window, forms the entrance to the inner ear. The middle ear optimizes the energy flow from air-borne sound in front of the ear drum to fluid motion in the inner ear by a mechanism called impedance transformation. One part of the impedance transformation is based on the lever ratio of about 1.5:1 produced by the different lengths of the arms of malleus and incus [162]. The lever ratio transforms oscillations of the ear drum with small forces into motions of the fluid with large forces. The law of energy conservation implies also that the tiny displacements of the ear drum are transformed into still smaller oscillations in fluid. An even larger transformation of the pressure is due to the ratio of the large ear drum to that of the small oval window. This ratio is about 17 [97]. Through the lever and area ratios, an almost perfect impedance match is reached in man in the middle frequency range

between 1 and 4 kHz. It allows optimization of the energy flow into the inner ear, which otherwise would be reflected.

The middle ear operates normally when it is filled with air at atmospheric pressure. The Eustachian tube, which connects the middle ear cavity to the upper throat, normally opens and closes periodically, thereby insuring that the static pressure in the middle ear will remain the same as atmospheric pressure. We experience a pressure difference when the Eustachian tube fails to open during ascent or descent in an elevator. For an elevation of 8 m, the change in atmospheric pressure is 100 Nm^{-2}, corresponding to a sound pressure of 130 dB relative to the 20 μNm^{-2} reference! This pressure causes a static deflection of the ear drum and increases the stiffness of the middle ear transmission system and sound transmission is attenuated.

B.1.3 Inner Ear (Cochlea)

The shape of the cochlea resembles that of a snail shell with two and one-half turns (in humans) and hence its name (Figure B.2). The central conical bony core of the cochlea is called the *modiolus*. The auditory nerve fibers run in this bone and exit the cochlea at its base. The outer wall of the modiolus forms the inner wall of a 30 mm long canal which spirals the full two and one-half turns around the central core.

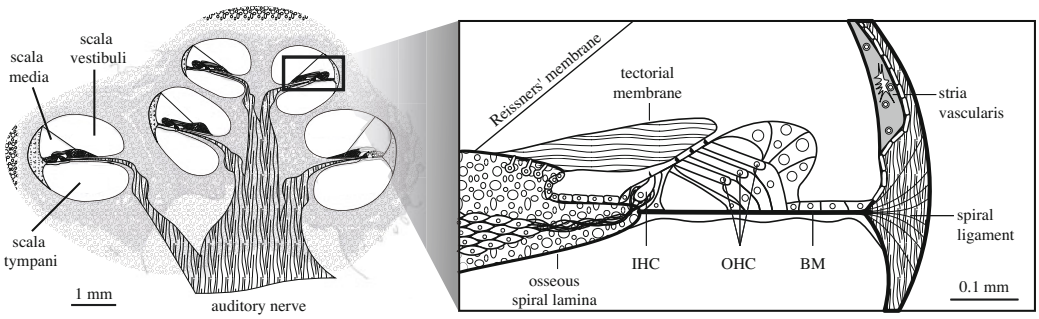

Fig. B.2. Section through the human cochlea (left) and magnified view of the organ of Corti (right). IHC: inner hair cell, OHC: outer hair cell, BM: basilar membrane. [1157], © Springer-Verlag Berlin Heidelberg 1999. With permission of Springer.

This canal is separated into three partitions called scales: Scala tympani is separated by the so-called cochlear partition, which is formed by a thin shelf of bone projecting from the modiolus (the ossesus spiral lamina), which is connected by the basilar membrane and the spiral ligament to the outer wall of the cochlea. The sensory organ of hearing, the organ of Corti, is located on top of the basilar membrane. As can be seen in the cross section of the cochlear spiral, the cochlear scalae become smaller and smaller in cross-sectional area as the apex is approached. Directly opposed, the basilar membrane becomes progressively wider towards the apex (Figure B.2). This is because the osseous spiral lamina is broadest at the cochlear base where the basilar membrane is only about 0.16 mm wide (in humans); at the apex the basilar membrane has broadened to about 0.52 mm.

Scala media and scala vestibuli are separated by a thin membrane, called Reissner's membrane (Figure B.2). At the apical end of the cochlea, scala vestibuli and scala tympani are connected by an opening in the cochlear partition, called helicotrema. Scala tympani and scala vestibuli are filled with perilymph, which resembles in its chemical composition other extracellular fluids. Perilymph is characterized by high sodium (Na^+) concentration of about $140 - 150$ mM and its low potassium (K^+) content of only around 5 mM. Scala media is filled with endolymph, which is unlike any other extracellular fluid found in the body. From its chemical composition, it resembles intracellular fluids. Its predominant cation is potassium with a concentration of about 157 mM; sodium is very low (1.3 mM). In addition to its special chemical composition, the endolymphatic space exhibits a considerable positive electrical potential within scala media of

about +80 mV relative to scala tympani and scala vestibuli, called the endocochlear potential. The chemical composition and the electrical potential of the endolymphatic space is sustained by active ion transport provided by the cellular layers of stria vascularis.

The organ of Corti lies just between the endolymphatic and perilymphatic spaces (Figure B.2). Its surface is sealed by tight junctions to keep the fluids separate. The basilar membrane, on which the organ of Corti rests, is composed mainly of extracellular matrix material with embedded fibers. In contrast to the tight surface of the organ of Corti, the basilar membrane is thought to be permeable to perilymph. In the organ of Corti, two types of sensory cells, one row of inner hair cells and three to four rows of outer hair cells are embedded. In humans, there are approximately 3,500 inner- and 12,000 outer hair cells. The membrane potential of inner hair cells is about -40 mV, of outer hair cells even as low as -70 mV. Both types display "hair bundles" or stereocilia, which project into the endolymphatic space. The stereocilia are arranged in several rows, which are graded in size. Stereocilia from different rows are connected by a fine filament, called tip-link. The current theory of transduction assumes that hair bundle deflection pulls on the tip-links, opening transducer channels which are close to the attachment points. This concept also explains that the hair bundle is only sensitive to mechanical stimulation in the direction of the tallest stereocilia. The transducer channel is a nonselective cation channel which is very impermeable to anions. Therefore, the transduction current is mainly carried by potassium (K^+) and calcium (Ca^{2+}) cations, driven by the large electrical potential between the endolymphatic space and the receptor cells. The driving potential sums up to 120 mV for inner hair cells and to 150 mV for outer hair cells. Potassium leaves the sensory cells via potassium channels present in the basolateral cell membrane and diffuses into scala tympani. It is interesting to notice that the intracellular concentration of potassium is as high as that of endolymph. Potassium is driven into the cells by the electrical potential and because of the concentration gradient, it can diffuse into scala tympani without requiring energy from the sensory cells.

The inner and outer pillar cells, the phalangeal processes of the Deiters' cells and the cylindrical bodies of the outer hair cells build a complex, three-dimensional truss (see Figure B.3). One peculiarity is that outer

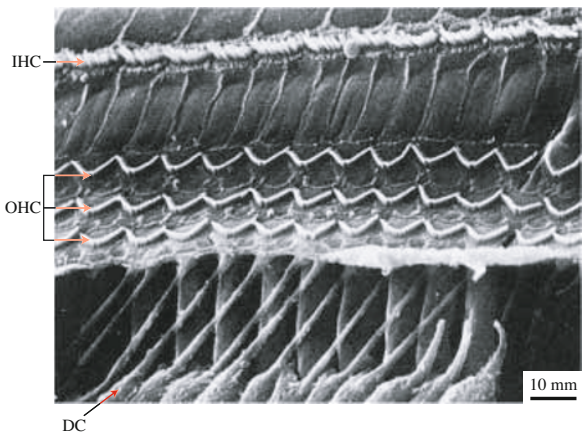

Fig. B.3. Scanning electron micrograph of the three-dimensional arrangement of the organ of Corti. IHC: inner hair cells, OHC: outer hair cells, DC: Deiters cells [301], with kind permisson of the authors..

hair cells are not in contact with other cells along their lateral surface but immersed in extracellular fluid. From Figure B.3 also the different morphology of the hair bundles becomes apparent: Whereas the bundles of the inner hair cells are arranged in a straight line, these of the three rows of outer hair cells are W-shaped.

The hair bundles of outer hair cells are excited by a shearing motion between the surface of the organ of Corti and the so-called tectorial membrane. The tectorial membrane is a gel-like structure composed of extracellular matrix material and it is in direct contact with the longest row of outer hair cell stereocilia. In contrast, the hair bundles of the inner hair cells seem not to be in direct contact with the tectorial membrane. Their bundles are probably driven by fluid forces. The transduction current flowing through the stereocilia is converted into a receptor potential in the cell body of the hair cell. At low frequencies, the receptor potential follows the stimulus cycle-by-cycle. Upon mechanical stimulation of the hair bundle in excitatory direction, tip-links are stretched and the transduction channels open. As positive K^+ ions are driven into the cell, the potential inside the cell becomes more positive. If the bundle is stimulated in the other direction, tip links relax and transduction channels close. However, as for inner hair cells at rest only 20% of transduction channels are open, the receptor potential is highly asymmetric when stimulated with a tone. On the basal pole of the inner hair cells about 10 to 30 afferent synapses are located, and upon depolarization of the cell membrane, voltage-sensitive Ca^{2+}-channels located in the basal pole of the cell membrane open, the increased Ca^{2+} level causes vesicles filled with transmitter to fuse with the lateral cell membrane and release transmitter into the synaptic cleft. This transmitter release triggers an action potential in the afferent nerve, and these electrical spikes transmitted to the brain finally lead to the hearing sensations.

These mechanisms work well at low frequencies, where the events of neural processing can easily follow the sound stimulus. Because of its refractory period, a single auditory nerve fiber can not respond to each successive cycle of a high-frequency sound. This problem can be partially overcome by the fact that more than one nerve fiber contacts a single inner hair cell. Each fiber is incapable of responding to every cycle of the stimulus, but collectively, they can do so. The temporal structure of the sound is conserved, as each fiber responds in the depolarized state of the inner hair cell. We call this behavior "phase locking". Even if the firing rate of a nerve fiber is too slow to follow the stimulus, basic features of its temporal structure—like the phase—are still coded in the neural pulse train. Still, the effect of phase locking is limited by several factors. The lateral membrane has a time constant of about 1 kHz, and above this frequency, the AC-amplitude of the receptor potential decreases. Also the synaptic processes like vesicle release and the generation of the postsynaptic potential are limited in speed and accuracy. This leads to a gradual loss of phase locking starting above 1 kHz, and above 3 kHz, phase locking is completely lost in humans.

In the frequency region, where the receptor potential can no longer follow the stimulus on a cycle-by-cycle basis, we see a depolarization of the inner hair cells membrane potential while stimulated. This so-called DC-component of the receptor potential follows the stimulus *envelope*. This is because of the asymmetry of the hair-bundle transduction. The depolarizing currents dominate upon sinusoidal stimulation and the receptor potential, low-pass filtered by the cell membrane, cause a depolarization of the cell membrane. All the information about the sound's frequency is lost in this signal, therefore, a different way to code high frequencies had to be developed by evolution.

This task was achieved by "sorting" sounds by frequency. Sound signals are mechanically preprocessed in a way that they are separated spatially. This concept is well-known as the frequency-place principle where high frequencies are located in the basal part of the cochlea only and low frequencies in the apical part. Different sets of auditory nerve fibers elicit different auditory sensations by virtue of their central connections. We will examine the mechanical frequency separation process in the sec Section B.1.4 and to do so, we will have to focus on the mechanical properties of the inner ear.

It is however not clear how specific excitations converge to yield well-defined pitch in general. There is increasing evidence that pitch is extracted in the time domain (periodicity analysis [580]). Moreover, pitch was found to be independent of the frequency-place mapping of the components of complex tones [581].

B.1.4 Cochlear Hydrodynamics: The Travelling Wave

The cochlea consists of three fluid-filled scalae, but from a mechanical point of view, the elastic properties of the thin Reissner membrane can be neglected compared to the stiffness of the basilar membrane. We can therefore simplify our mechanical investigations to a fluid model with two chambers separated by the cochlear partition (compare Figure B.4). The basilar membrane contributes a large part of the elastic properties of

the partition. Its width is increasing from base to apex from about 0.16 μm to 0.52 μm (in humans). Its thickness, on the other hand, decreases along the cochlea. Thus, its stiffness decreases greatly along the length of the cochlea.

At the basal end of the cochlea there are two openings to the cochlear ducts, one on each side of the cochlear partition, that are covered by membranes. One is called the oval window and, as we already mentioned, it is in contact with the stapes foot plate. The other, the round window, is just below the oval window. Because we can assume the cochlear fluids as incompressible, the round window has to move out of phase if the oval window is driven by stapes motion.

If we consider very slow stapes motions, the fluid is pushed along the entire length of scala vestibuli, through the helicotrema and back along scala tympani. Thus, the helicotrema provides a low-frequency shunt for extremely low frequencies. When the stapes moves into the cochlea, pressure builds up in the fluid, which deflects the cochlear partition. Pushing fluid requires overcoming inertial forces generated by the fluid mass. The elastic properties of the basilar membrane in combination with the mass of the surrounding fluid constitutes second-order resonators. As the stiffness of the basilar membrane is very high at the basal end of the cochlea and much lower at its apical extreme, the resonant frequency of the cochlear partition monotonically decreases from base to apex. Inversely, the time constant of each resonator increases from base to apex. However, the basilar membrane is not under tension, and it does not respond like a series of *independent* resonators, like the strings of a harp. Instead, each part of the cochlear partition is coupled to the next by the cochlear fluids, and due to the large inherent friction, they are highly damped. For a periodical motion of the stapes, the cochlear partition is first set into motion at the basal extreme, where the mechanical time-constant is smallest. Because the stiffness is very high, the deflection of the basilar membrane is fairly small. The deflection propagates in the form of a wave in apical direction. As the time-constants of the partition increase and the stiffness decreases, the response will be more and more delayed but its amplitude will grow. At the location of resonance, the wave will reach its maximum and lose its energy, its amplitude will drop very rapidly. The location of "cochlear resonance", the place where the displacement—and therefore the excitation—of the cochlear partition reaches its maximum, depends on the stimulus frequency. Low frequencies will travel along the basilar membrane and reach a maximum close to the cochlear apex, high frequency sounds will exhibit their maximum response close to the cochlea base and fade out.

The exact form of the vibration response of the cochlear partition was investigated by Georg von Békésy in human cadaver ears [97], a feat which earned him the Nobel Prize. He found that the deformation of the basilar membrane is a traveling wave. The wave starts at the cochlear base, where the basilar membrane is stiffest. It propagates toward the apex with a time delay that depends upon its own mechanical properties and the properties of the surrounding fluid. Its vibration amplitude is increasing until it reaches a maximum, close to the location of cochlear resonance, and from then on, the wave diminishes rapidly. Because the stiffness gradient of the basilar membrane is approximately logarithmic, the peaks of the excitation patterns of sinoidal tones are located on the basilar membrane with a logarithmic frequency spacing. Figure B.4 schematically illustrates traveling waves elicited by a stimulus composed of three frequencies, together with the envelope of the peak displacement. The peaks of the three waves are clearly separated along the cochlea, however, there is also considerable overlap between the waves and recordings from the auditory nerve have been found to be much more frequency selective, especially close to threshold, than the mechanical responses observed in cadaver ears. Therefore, a second mechanism is required to boost the frequency selectivity of the vibration responses.

B.1.5 Active Amplification of the Traveling Wave Motion

We have so far neglected the function of the outer hair cells, the second group of receptor cells within the organ of Corti. Despite the at about three to four times higher number of outer hair cells, only 5% of the afferent fibers innervate outer hair cells. The fibers are so-called type II fibers, they are highly branched and each fiber innervates dozens of outer hair cells. Little is known about these fibers because they are small and unmyelinated, making it difficult to record their activity. They are expected to be less sharply tuned, since they innervate a broad region of the cochlea, and, because of their lack of myelination, their conduction

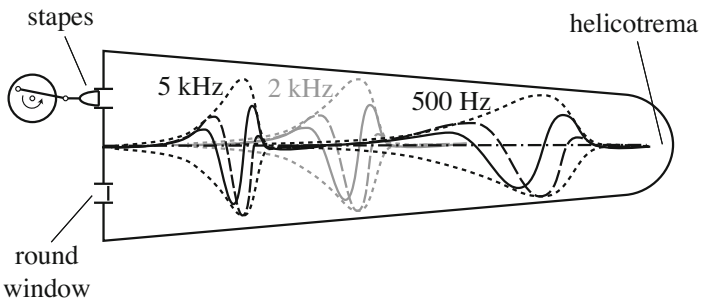

Fig. B.4. Schematic illustration of the traveling waves elicited by three pure tones with frequencies of 500 Hz, 2 kHz and 5 kHz. The displacement of the basilar membrane is shown at the instant T_0 (solid line) and a quarter of a cycle later (dashed line). The dotted line indicates the envelope of the wave.

velocity is likely to be very slow. Most of the frequency selective and time-critical auditory information must therefore be carried by the afferent fibers originating from the inner hair cells. Whereas inner hair cells are not directly innervated by the efferent system, myelinated fibers from the medial to the medial superior olivary nuclei make direct synaptic contact with the outer hair cells. Outer hair cells seem to be under neural control, much more like muscles than sensory cells. A final indication of their "active" role was established when their ability to change their length upon electrical stimulation was detected. Modern research assumes that the outer hair cells indeed provide an active, mechanical amplification of the cochlear traveling wave. As neural processing is by far too slow to keep up with high frequency hearing, current concepts assume that amplification relies on a local mechanical feed-back process [239]:

> Outer hair cells sense motions of the cochlear partition by converting shearing motion between the surface of the organ of Corti and the tectorial membrane into an electrical receptor potential. Upon depolarization, the outer hair cell reacts with a contractile force, which is fed back into the motion of the basilar membrane.

From theoretical calculations derived from measurements in the inner ear, it is required that energy is pumped into the vibration of the cochlear partition in a region starting before the traveling wave reaches its maximum up to the place of cochlear resonance. The function of this amplification process is still unclear in its details, but from the observations of active cochlear mechanics it has wide reaching consequences. Measurements in the basal part of the cochlea indicate, that the amplification boosts cochlear sensitivity up to thousand-fold. The amplification is limited to a narrow frequency range, covering only about half an octave. The active traveling-wave response becomes very sharp in the region of cochlear resonance. The amplification of the vibration response is required to achieve the extraordinary sensitivity of the hearing system.

The second hallmark of amplification is non-linearity. The amplification process boosts only weak sounds, it saturates at increasing levels. This non-linearity greatly compresses the dynamic range of the mechanical responses. This is important, because the dynamic range of the inner hair cell receptor potential is limited to a range certainly not exceeding 60 dB. This non-linearity also has unwanted side-effects: If two sinoidal tones are presented, the non-linearities of the hearing-organ generate so-called distortion products, additional tones which we perceive under certain circumstances. In general, however, by virtue of its construction, artifacts due to the non-linearities of the inner ear are surprisingly small.

Figure B.5 shows the excitation pattern expected on the human basilar membrane when stimulated with a 3 kHz sinoidal tone. For high sound levels (i.e., 100 dB), the feed-back amplifier is saturated and the traveling-wave is almost purely passive. It is highly damped and its envelope shows the characteristic shallow

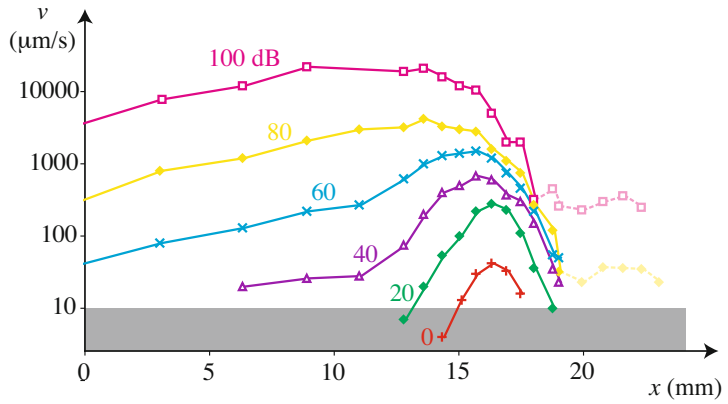

Fig. B.5. Reconstruction of the excitation pattern in the human cochlea for a 3 kHz sinoidal tone. Original measurements were recorded in a chinchilla cochlea at a distance of 3.5 mm from its most basal extreme [781]. The characteristic frequency of this location was 9.5 kHz. Data has been converted assuming a frequency-place map of 8 mm/octave to illustrate the excitation pattern in a human cochlea. The shaded area indicates excitation below neural threshold, which is expected between 50 $\mu msec^{-1}$ and 1 nm.

increase from base to apex and a sharp decay after the maximum is reached.[1] The threshold of the auditory nerve is somewhere between a minimal basilar-membrane velocity of 50 $\mu msec^{-1}$ and a displacement of 1 nm, exact values are still unknown. The broad traveling wave at high levels indicates that a large number of nerve fibers are stimulated, especially in the basal part of the cochlea. For faint sounds, the traveling-wave response becomes sharper and sharper and its envelope is almost symmetrical for levels below 40 dB. The location of maximum amplitude for a sinoidal tone at low levels is called its characteristic place. Only nerve fibers originating from a very narrow region around the characteristic place of the cochlea are stimulated. Note that for increasing levels, the maximum of the traveling wave shifts considerably in the basal direction. If we analyze the level-dependence of the responses at various locations of the cochlea, we clearly see the effects of the non-linearities of the amplification. At the characteristic place (about 16 mm in Figure B.5), the velocity amplitude increases from a value of 42 $\mu msec^{-1}$ at 0 dB to 5 $msec^{-1}$ at 100 dB. Without amplification, the amplitude of the traveling-wave response would be expected to drop by a factor of 10^5, or from 5 $mmsec^{-1}$ to 50 $nmsec^{-1}$ from 100 dB to 0 dB! The amplification therefore is almost a factor of 1000 or 60 dB. In addition to enabling the detection of weak signals, the amplification therefore also compresses the dynamic range, again by almost 60 dB. The inner hair cells, at the location of the characteristic place, have to cope only with a stimulus ratio of a little bit more than 40 dB to cover a 100 dB-level change of the sound stimulus.

B.1.6 Neural Processing

The difficulty to understand neural processing of sound strongly stems from the extremely complex innervation from the auditory nerve to primary auditory cortex (Heschl's gyri) in the temporal lobe (see Figure B.6). We have to stress that the image in this figure is considerably simplified, and that in particular, there are also connections from the cochlea to the ipsilateral auditory cortex, as well as efferent nerves from the auditory cortex down to the cochlea.

[1] The data in Figure B.5 (dashed lines) indicates that further towards the apex, the wave does not die out completely in this experiment. It is still under debate, whether this remaining response is also present in the intact human cochlea.

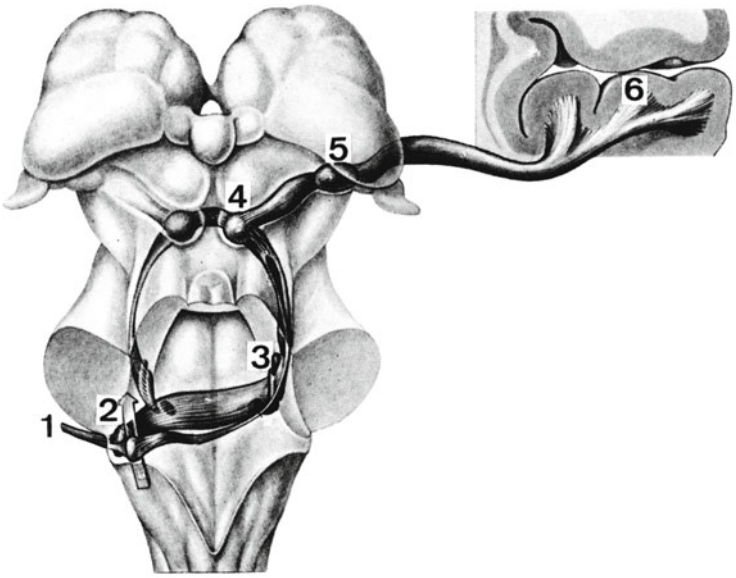

Fig. B.6. (Private archive H. G. Wieser and G. Mazzola) This simplified image shows the six relevant relais stations of the auditory path from the cochlea to the Heschl gyri: (1) Nervus cochlearis, (2) nucleus cochlearis, (3) nuclei superiores olivae, (4) colliculus inferior, (5) corpus geniculatum mediale, (6) gyri Heschl.

The auditory system includes at least five "relays stations", whence it is clear that any particular functional decomposition (like Fourier's) will not be transferred unchanged to the auditory cortex. For example, the tonotopy of the latter is a multiply distributed one, see Figure B.7.

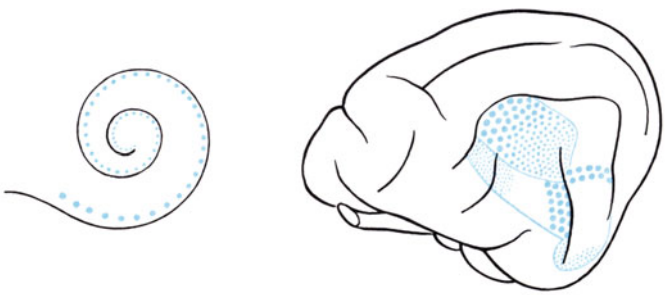

Fig. B.7. (With kind permission of the Thieme-Verlag, [516]) Left: tonotopy in the spiral of the cochlea: high pitches are thick points, low pitches are thin points. Right: Corresponding multiple tonotopy in the auditory cortex of the cat.

By tonotopy, one understands the spatial distribution of excitation patterns according to specific pitch. More recent research also demonstrates that outside the auditory cortex, i.e., in the limbic system (the hippocampal formation, to be precise, which plays an important role for emotional and memory tasks), one finds a refined processing of pitch information [678, 679, 1122, 1123].

The "template fitting model" of Julius Goldstein [378] shows how far we are from understanding the neural pitch processing. In this model, the mathematical principle of a neural "central pitch detector" is proposed, from which the fundamental frequency of a periodic wave can be extracted if its Fourier components are known. The central pitch detector is however charged with the solution of local minima problems for functions in two variables—a rather heroic task for a small neural population. It seems hopeless to identify

within the neural network the physical realization of a dynamic system that solves these differential conditions. We presently have no chance to verify the model physiologically, since the human ethics excludes adequate experiments in humans.

In view of these facts, it is not only logically erroneous and experimentally very delicate to infer the higher sound processing from the superficial auditory physiology of the ears. And ethically, such an attempt is problematic since one runs the risk to "justify" discrimination of "degenerate music" against the acceptance of so-called "commonly accepted" music. It is also not clear how much such investigations reveal grown and trained configurations instead of biological inheritance.

B.2 Discriminating Tones: Werner Meyer-Eppler's Valence Theory

From our everyday experience and from specific experiments it follows that we do not really hear the single tone events or chords, motives or rhythms as they have been parametrized on the level of physical or mathematical description. In fact, despite the very sensitive physiology of Corti's organ and its hair cells, we cannot distinguish all physically or mathematically distinct sound objects. For instance, sounds with frequency above $20,000\ Hz$ or below $0\ dB$ are indistinguishable since you cannot hear them. Or two sounds p, q with a phase shift, i.e., $p(t + \Delta) = q(t)$ are indistinguishable. More important is that even under ideal condition we cannot distinguish sounds with arbitrary precision. Every singer or violinist who has to adapt his/her pitch to a prescribed context knows this. Investigations on variations of instrumental intonation show a remarkable bandwidth [117]. The same is valid for listening to time values, loudness degrees and instrumental colors.

It is mandatory but not easy to take into account these phenomena. Werner Meyer-Eppler [735, 736] has attempted to solve the problem by use of the concept of a "valence". According to this approach,

Definition 127 *In a specific context, given two sound objects s_1, s_2, s_1 is* metamere *to s_2 with regard to a given predicate P (short: P-metamere, in symbols: $s_1 \sim_P s_2$) iff the s_1 cannot be distinguished from s_2 with respect to P by human listeners. The set $\sim_P s$ of sound objects which are P-metamere to a given sound s are called the* valence of s.

Since usually the relation $\sim_P$ is symmetric, one calls two sound objects s_1, s_2 metamere if $s_1 \sim_P s_2$. If a sound object s is defined by a sequence $P_1, \dots P_k$ of predicates, one defines its valence as being the sequence of valences $\sim_{P_1}, \dots \sim_{P_k}$, and we may then reduce the total valence to those components which do not include all sound objects since these predicates are not relevant to the distinction of sound objects. The union $\bigcup_{\text{supporting valences}} \sim_{P_{i_j}}$ of supporting valences is in fact the valence of the conjunction predicate $\bigwedge_i P_i$. Predicates which are relevant in this sense are called *valence supporting* by Meyer-Eppler.

In practice, if we are given a sound whose sound color is described by partials with frequencies up to the limit frequency $f_0\ Hz$, and which last $d\ sec$, Meyer-Eppler deduces a maximal number of numerical predicates (dimensions) that are relevant to the valence, i.e., of valence supporting numerical predicates. This limit is called the *maximal structure content*, and its value is $K = 2.d.f_0$. It is however an open fundamental question of sound color theory, which and how many valence supporting predicates must be chosen in order to yield a differentiated perception of sound colors. Probably, these valences define a multiply connected topological space in the physical parameter space, but see also Section 12.3.

Although Meyer-Eppler's conceptualization is plausible, it hides two delicate problems. The first concerns the context where valences take place, as related to the predicate in question. If the context's specification is neglected, the valence concept loses its meaning. Let us make two representative examples concerning pitch and onset. To begin with, we have to agree on who is accepted as a listener. In the sense of a statistical approach, Meyer-Eppler proposes that two pitches should be called metamere if at least 90% of the test subjects cannot distinguish them [736]. Moreover, one has to agree on the parameter(relations) of the test sounds. The result will depend essentially on the choice of instruments, flute, brass etc., and conditions upon the duration and the onset distances of test sounds. For example, the simultaneous presentation of sounds

of several seconds duration will yield smaller valences because of beat effects, as compared to a comparison of non-overlapping sound events.

Besides the parametric conditions, the context can also depend on the chosen music. Let us discuss this on the onset parameter, for example. In [736], the duration valence density is indicated by $50 - 60$ per second. This means that sounds which are less than $1/50 - 1/60$ *sec* apart are perceived as being simultaneous. However, the musical context of such a claim is relevant. Within a very slow piece with a small number of instruments, the temporal variation of $1/8$ *sec* will scarcely be noticed, whereas in a rhythmically very dense and fast piece, $1/60$ *sec* is known to define a quite coarse grid.

The second problem is important for the theoretical significance of the valence concept. It relates to the fact that metamery is not an equivalence relation[2] in general. More precisely, any of the wanted properties: reflexivity, symmetry, and transitivity, can be violated. Reflexivity can be violated if the comparison of two sounds is temporally so separated, by several hours, say, that the human memory fails to recognize one and the same sound. Symmetry is violated if the order of appearance of sound objects is relevant, for example, a very loud sound, immediately followed by a very soft one, can mask the latter's properties. The most dramatic failure is the absence of transitivity: $s \sim_P t, t \sim_P u$ does not always imply $s \sim_P u$. For example, if three pitches are such that the first is perceived as being equal to the second, and the second being equal to the third pitch, this is not entailed by equality of first and third pitch! Therefore, pitch valences are not equivalence classes, they may overlap. This means that the attempt to define pitch by an esthesic position in music psychology must fail. The perceptional concept of pitch is a non-transitive relation among tones, and therefore is not an attribute of tones. You hear that two tones have the same pitch, but you do not hear the pitch. Therefore corresponding attempts such as [784] must fail.

This has important consequences for musical practice and for theoretical aspects. In practice it is desirable to select grids of sounds such that their valences would not overlap. This needn't be a grid which is fixed once for ever, it might be a time-dependent construction, but it must yield a locally disjoint valence set.

With the common notation, such a grid is realized as an orientation device as well as an acoustic and performative scheme. The continuum of onsets, durations, pitches, and sound pressure values is quantized in the well-known way such that Boulez' "notched tone space" can be taken as a grid behind the *realiter* played or heard.

The reality layer of a grid is a mental or psychological one. The blurredness of hearing, as it is expressed in valences, has to be subjected to a cognitive interpretation. Semiologically speaking, the valence is the significant, the expression for a meaning which relates to our understanding of music. By use of the grid which is superposed to the valence perception, it is possible to associate the valences of perceived sound to objects of our imagination. The psychological quality of pitch then results from this mapping as a grid object in our imagination. The semiotic power of this signification depends upon the definition of the actual grid.

A music-theoretically fundamental grid is the selection of the pitch arsenal, the tuning, wherein tones may be played. Let us first look at the chromatic w-tempered scales (see Section 7.2.1.3). If w is not too large, valences of neighboring tones can be separated. From our experience with microtonal music, $w = 36$, i.e., tempered sixth-tone intervals, is not too large. So the postulate of an adjustment of pitch by inner grids is acceptable for w-tempered scales with $w \leqslant 36$.

The situation of just intonation (see Section 7.2.1.4 and Appendix A.2.3) is much more delicate. For each pitch $H(x_0)$ of a point x_0 in the Euler grid $\mathbb{Z}^3$ of just intonation, and for any positive real number ϵ, there are infinitely many points x such that $|H(x - x_0)| < \epsilon$, see Proposition 76 in Appendix A.2.3. In particular, infinitely many pitches fall into the valence of x_0. *In the valence semiology, just intonation is infinitely homonymic.* This problem can be solved by a restriction of the context, where just intonation music is played. As soon as the local context is a small region of the Euler space, for example, a small neighborhood of a determined finite portion of the chromatic scale, valences can be used to distinguish just grid points.

A second difficulty for just intonation relates to the perception of pitch differences. A classical argument of just music theory is Euler's substitution theory [308], according to which intervals are heard in a way such that the frequency relations of their tones form fractions a/b (in reduced representation) with minimal

[2] See Appendix C.2.

numerator a and denominator b. This is the basis of the classical consonance-dissonance theory which we address in the following section. In the grid of just intonation this means that the pitch difference of the interval is corrected/adapted to an interval which is realized by two points of the just intonation grid under the constraint that their (Euclidean or 1-norm) distance is minimal. Of course, this correction has to happen within a valence in order to be a reasonable process of human hearing. However, it is easy to see that in general, there are several solutions $a/b, a'/b', a"/b", \ldots$ with minimal distance. So Euler's substitution theory would have to impose a contextual restriction for single tones as well as for intervals.

These considerations should be taken for nothing more than they are: esthesic aspects of hearing. Nothing prevents us from doing music theory on neutral and poietic layers without bothering about valences. But then, one has to be conscious of the fact that highly differentiated mathematical structures may be blurred by the semiology of valences from auditory psychology and physiology.

B.3 Symbolic, Physiological, and Psychological Aspects of Consonance and Dissonance

For a long time, mathematical reflections in music were centered around the problematic concept couple of consonance and dissonance. This is based on the ancient Greek Pythagorean tradition where consonance and dissonance of intervals was laid in the involved frequency relation. Perfect consonant intervals corresponded to ratios like 2/1 for the octave, 3/2 for the fifth, and 4/3 for the fourth. This simple arithmetic corresponded to the philosophy of the metaphysical tetractys. See [672, 784] for a historical discussion of these roots.

Here, we simply want to recall that a unified mathematical foundation of musical thinking in the paradigm of simple consonant frequency ratios could not survive the differentiated development of theories in the contrapuntal setting [924], the psychological foundation of musical relations as introduced by René Descartes [262], and the discovery of physical partials by Marin Mersenne [630].

According to these more recent positions, the problem of consonances and dissonances changes as a function of the layer of reality where it is investigated—and on each layer it is not a minor one. In the present shorthand presentation, this result seems to be a provocative one since the conceptual unity seems violated. We want to make clear that we really are dealing with three different meanings of the sonance concept—Euler's gradus suavitatis on the mental layer, Helmholtz' beat model on the physical layer, and Plomp-Levelt's psychometrics on the psychological layer. In itself, each of these approaches is consistent, the problem only arises if one attempts to reduce one reality to another one. Following the knowledge about the neural processing of sounds (see the previous discussion in this chapter), it is hardly astonishing that psychological and physiological layers are not congruent: what the ear (in Helmholtz' model) does not "like" can very well be "agreeable" for the limbic system or the auditory cortex.

B.3.1 Euler's Gradus Function

Being a number theorist, Euler was interested in prime numbers. A priori, his gradus function Γ [307] is a purely number-theoretic function, it is defined as follows: According to the prime factorization of integers (see Appendix D.2), a positive integer a is the unique product $a = p_1^{e_1}.p_2^{e_2}.\ldots.p_n^{e_n}$ of positive powers of primes $p_1 < p_2 < \ldots p_n$ (with the singular case of zero factors and a product = 1 for $a = 1$). Euler's formula, the *gradus suavitatis*, is $\Gamma(a) = 1 + \sum_{1 \leqslant k \leqslant n} e_k(p_k - 1)$, and more generally $\Gamma(x/y) = \Gamma(x.y)$ for a reduced fraction x/y.

In just intonation—where Euler's approach belongs—only intervals of tones are considered whose frequency ratio is a positive rational number x/y. The gradus function is defined for such intervals. Each interval evaluates to a positive integer. The frequency ratios with Γ-values $\leqslant 10$ are listed in table Appendix L, see also Figure B.8.

In just intonation, intervals can be read as differences $\Delta = x - y = (e, f, g)$ of Euler points x and y in the grid $\mathbb{Z}^3$. For the gradus function, we may restrict to non-negative coordinates e, f, g. We then have

$$\Gamma(\Delta) = \Gamma(2^e.3^f.5^g) = 1 = 1 + (2-1)e + (3-1)f + (5-1)g \tag{B.1}$$

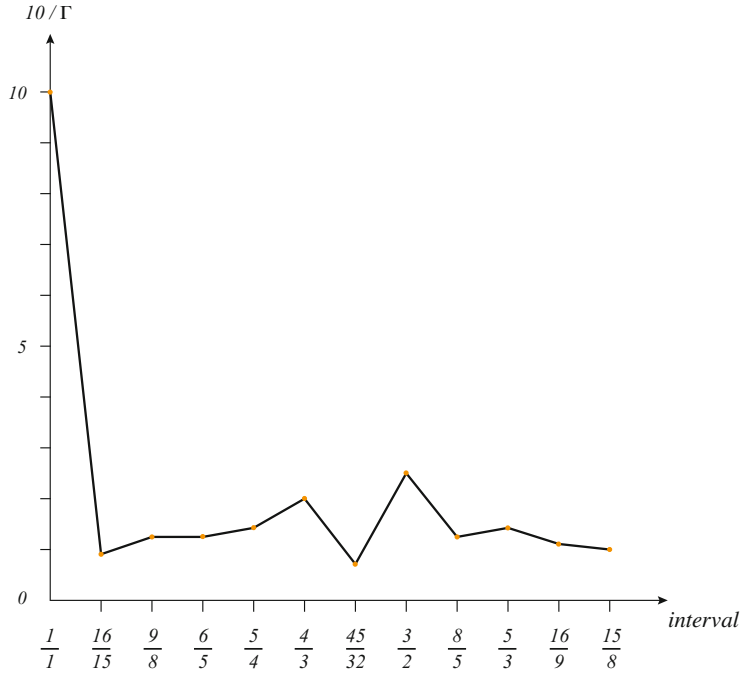

Fig. B.8. The agreeableness of intervals within the octave in just intonation according to Vogel's chromatic [1089], see Section 7.2.1.4, is represented. It is reasonable to represent the reciprocal values $10/\Gamma$ instead of Γ since the latter is rather a 'gradus dissuavitatis': small Γ values are taken for the octave and the fifth, large ones for the second and tritone. The factor 10 is only a scaling constant. The order of Euler's valuation is this: Prime, fifth, fourth, major third/major sixth, minor third/minor sixth/major second, minor seventh, major seventh, minor second, tritone.

or as a scalar product

$$\Gamma(\Delta) = 1 + (\Phi, \Delta) \text{ with } \Phi = (1, 2, 4). \tag{B.2}$$

If we compare this formula with the pitch formula

$$H(\Delta) = (H_{prime}, \Delta)$$

where H_{prime} is the prime vector from Section A.2.3, then we observe a similar construction. The gradus function is something like a pitch function, but the 'direction' is Φ instead of H_{prime}, see Figure B.9. The ranking of intervals by the gradus function was already criticized by Euler's contemporaries Mattheson, Mitzler, and Rameau [161]. But it is the merit of Euler to have defined a ranking by a linear expression which, together with pitch and octave coordinate defines a coordinate system for the Euler space on the one hand, and considers the consonances and dissonance, on the other. The disadvantage of Euler's approach is that it is based on the valence-theoretically invalid substitution hypothesis (see Section B.2).

B.3.2 von Helmholtz' Beat Model

Hermann von Helmholtz proceeds from the hypothesis that beats between partials of two tones is responsible for sonance phenomena. The fact that he uses partials is bound to Ohm's postulate that we have a cochlear Fourier analysis [457]. Hence Helmholtz' approach only regards the cochlear basis of music perception and not the higher limbic and cortical auditory processing. A *beat* is the periodic amplitude variation which results from the superposition of two sinoidal waves which have a frequency difference $\Delta = f - g$, the *beat frequency*[3] which is small with respect to their frequencies f, g.

[3] One uses the trigonometric equation $\sin(x) + \sin(y) = 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})$, see Figure B.10.

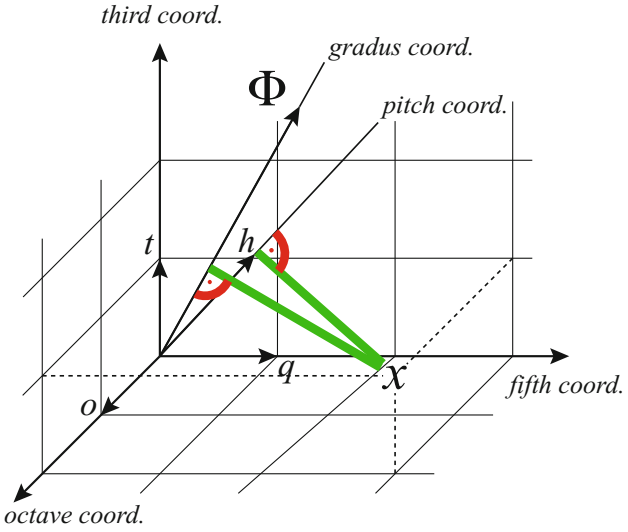

Fig. B.9. The gradus function is something like a pitch function, but the 'direction' is Φ instead of H_{prime}, Both values are obtained from a scalar product of an Euler point x (of a difference point Δ, respectively) with H_{prime} (with Φ, respectively). However, the point is not uniquely determined by the 'gradus coordinate', in contrast to the 'pitch coordinate'.

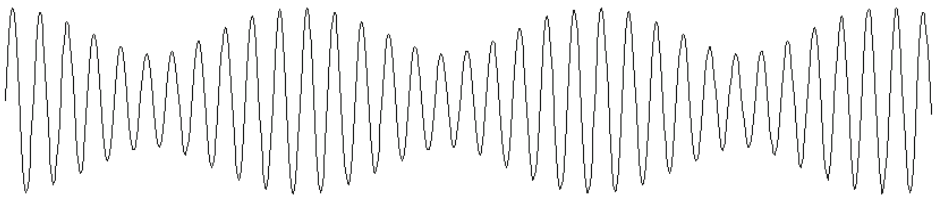

Fig. B.10. A beat between two superposed sinoidal waves is characterized by a periodic amplitude variation whose frequency is twice (!) the difference of the given frequencies.

Helmholtz calculates the *roughness*, i.e., the degree of dissonance of an interval which consists of two tones p, q as the sum of the *beat intensities* $I_{n,m}$, which are associated with the nth partial of p and the mth partial of q, where $I_{n,m}$ is supposed to have a strong maximum for beat frequency $\Delta_{n,m} = 33\ Hz$.

Therefore Helmholtz' dissonance concept depends on the pitches and the involved sound colors. On the example of the violin, Helmholtz obtained good coincidences with Euler's gradus function. This model is impressive since it explains the experience according to which consonance is a function of the instrument and the absolute pitch of the interval tones. Its experimental verification is somewhat problematic. It essentially depends on the measurability of the beat intensities in the cochlea. Since non-linear distortions on the sound's way to the cochlea change spectra, one would have to perform ethically problematic invasive cochlear measurements. Moreover, individual statistical variations of the non-linear distortions would make the experiments even less robust.

A fundamental doubt on the model's validity results from binaural experiments in hearing by Heinrich Husmann [494], where the interval tones are presented on separate left and right headphone inputs. In this case, no beats can intervene in the cochlea. Nonetheless, the experiments also revealed consonance as "happy moments of within the general (interval) disaster". The hypothesis that in the binaural experiments,

Helmholtz beats must occur in the relays station of the corpus geniculatum mediale (see Figure B.6) is speculative and demonstrates the limit of physiological models.

B.3.3 Psychometric Investigations by Plomp and Levelt

In the psychological reality of interval perception, the judgment of interval qualities looks quite different. In their investigation of "pleasantness" of intervals, Reiner Plomp and Wilhelm Levelt [850] have presented pairs of sinoidal tones and asked for pleasantness as a function of the given interval. The experiment was intendedly performed with musically untrained individuals in order to avoid judgments as a function of musical knowledge. Figure B.11 shows the resulting valuation curve. It is quite different from Euler's function as shown in Figure B.9.

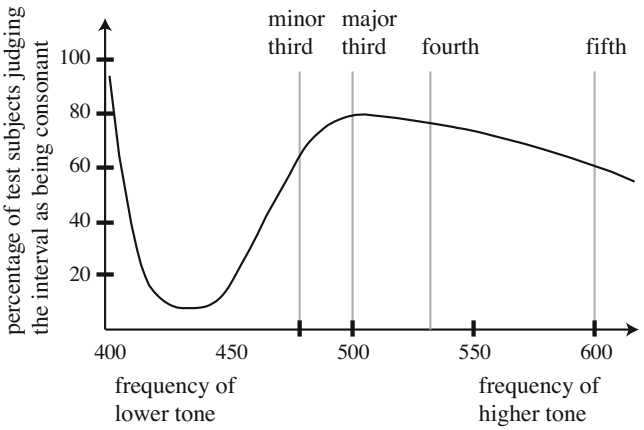

Fig. B.11. (From [914], with permission of Springer-Verlag) The psychometric investigation [850] of Plomp and Levelt yields a valuation which differs significantly from Euler's Γ function and is based on sinoidal tones shown in Figure B.9.

Using this curve, Plomp and Levelt have tried to infer a description of Helmholtz' beat intensities, a procedure which was already recognized as being problematic in Appendix B.3.2.

B.3.4 Counterpoint

There is however another—quite remarkable—point of view of the consonance-dissonance phenomenon, which has been poorly recognized within the psychoacoustic discussion, namely the prominent meaning of the concept pairing in the contrapuntal tradition, which was elaborated in the High and Late and Middle Ages, and which was encoded in an exemplary way by Johann Joseph Fux' *Gradus ad Parnassum* [349]. Carl Dahlhaus [230] has rightly pointed out that the textural function of the contrapuntal consonance concept is not yet fully understood. Interestingly, in the framework of the core theory of counterpoint, the interval of the fourth is dissonant, in contradiction to the other theories. It is inconceivable that the mathematical and physiological theories of counterpoint never included this perspective. A mathematical model of counterpoint is discussed in Chapters 29 through 31.

B.3.5 Consonance and Dissonance: A Conceptual Field

Even on the mental level, the concept of consonance and dissonance is multiply explained, without necessarily leading to contradictions. In fact, Euler's approach was a neutral mental (number-theoretic) one, whereas the

contrapuntal approach is poietic. If we look at the status quo of the consonance-dissonance discussion and the fight for a valid final semantics, we are confronted with a disaster. The mathematically arbitrary ornaments which composers such as Klarenz Barlow [82] or neo-Pythagoreans such as Martin Vogel [1091] add to Euler's formula does not interest psychoacousticians such as Ernst Terhardt [1050], who would extrapolate cochlear findings into the auditory cortex, and the latter approach cannot shed light onto the esthetics of music.

What is common to all these positions adds to a concept of consonance and dissonance which is a conceptual field within the topography of music, a field with a quite ubiquitous presence. As a musical thought it results from a fundamental linear dynamic between polar extremals. It should be a main task of mathematical music theory to elaborate reliable and semantically reasonable, but esthetically undogmatic, models for such a way of thinking music.

Part XXI

Appendix: Mathematical Basics

C

Sets, Relations, Monoids, Groups

C.1 Sets

The language of sets describes mathematical facts in a classical way. An alternative foundation to sets is the language of categories, see Appendix G.

A *set* M is an object which is defined as a collection of uniquely determined objects which are also sets. These objects are called the *elements* or *points* of M. Two sets are equal iff they have the same elements. Whenever we say that M is a set, we mean that it is defined in a consistent way, i.e., without causing any contradiction. Existence of a mathematical object means that the object's definition causes no contradiction in classical logic (A is identical to A, (exclusive) either A or non A, and exclusion of a third). Then, for any set m it is either an element of M or it is not. One writes $m \in M$ for "m is an element of M", or also "m is a point of M". In order to define a set M by its elements $m, m' \ldots$, one also writes $M = \{m, m', \ldots\}$. Observe that multiple enumeration does not change the set, for example $\{x, x, x, y\} = \{x, y\}$.

A set N whose elements are all elements of M is called a subset of M, in signs: $N \subseteq M$, also sometimes $N \subset M$ iff $N \subseteq M$ and $N \neq M$. Two sets are equal iff they are mutually subsets of each other. The *empty set* $\varnothing$ is defined as having no elements. It is a subset of any set. A set is called *finite* if is empty or its elements can be indexed[1] by a sequence[2] $0, 1, 2, 3, \ldots n$ of natural numbers. Otherwise it is called *infinite*.

C.1.1 Examples of Sets

Example 92 $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \pm n, \ldots\}$, the set of *integers*; $\mathbb{N} = \{n \in \mathbb{Z}, n \geqslant 0\}$, the set of *natural numbers*; the set $\mathbb{Q} = \{p/q, p, q \in \mathbb{Z}, q \neq 0\}$ of *rational numbers*; the set $\mathbb{R}$ of *decimal or real numbers*, e.g., $x = -741.76, \pi = 3.1415926 \ldots$. We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, where integers p are identified with rational numbers of form $p/1$, whereas rational numbers are identified with periodic real numbers.

Example 93 If M, N are sets, their *difference* $M - N$ or *the complement of* N *in* M is the set of points of M which are not in N. If $V = (M_i)_i$ is a family of sets, we denote by $\bigcup V$ the *union of* V, whose elements

[1] A mathematically correct definition of finiteness is this: A set if finite iff it is not in bijection with any proper subset.

[2] Recall that in this book, we make the logical, though not very common usage of the ellipsis symbol "...": it means that one has started with a sequence of symbol combinations which follows an evident law, such as $1, 2, \ldots n$, or $a_1 + a_2 + \ldots a_n$. The evidence is built upon the starting unit, such as "1," or "a_1+" in our examples, and then the following unit, such as "2," or "a_2+", and then inducing the following units to be denoted, such as "3," or "a_3+", "4," or "a_4+", etc., until the sequence is terminated by the last symbol, such as "n" or "a_n" in our examples. The ellipsis means that the building law is repeated, and as such, it is a meta-sign referring to the inductive offset. Therefore the more common notation $1, 2, \ldots, n$, or $a_1 + a_2 + \ldots + a_n$ is not correct. In the limit, for $n = 3$, it would imply a notation such as $1, 2, , 3$ or $a_1 + a_2 + +a_3$, which is nonsense. Moreover, in complicated indexing situation, the common notation would be overloaded.

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https://doi.org/10.1007/978-3-319-64495-0_3

are precisely the elements collected from any of the M_i, for finite families $V = M_1, \ldots M_n$, one also writes $\bigcup V = M_1 \cup \ldots M_n$. In particular, $\bigcup \varnothing = \varnothing$. A covering of a set X is a family V such that $\bigcup V = X$. The *intersection of a family* V is the set $\bigcap V$ consisting exactly of those points which are points in any of the family's member M_i. For finite families $V = M_1, \ldots M_n$, one also writes $\bigcap V = M_1 \cap \ldots M_n$. In particular, $\bigcap \varnothing = AllSet$, the set whose elements are all existing sets. In less universal contexts, one only takes the intersection with regard to a large superset of the family's members. Observe that there is no reason why *AllSet* should not exist, however, not any of its subcollections defined by predicates equally exists. For example, the subset of all sets not containing themselves as elements does not.

A *partition of a non-empty set* X is a covering such that any two of its members are non-empty and *disjoint*, i.e., intersect in the empty set.

C.2 Relations

Definition 128 *If x, y are two sets, the ordered pair (x, y) is defined to be the set*

$$(x, y) = \begin{cases} \{\{x\}\} \text{ if } x = y, \\ \{\{x\}, \{x, y\}\} \text{ else.} \end{cases} \tag{C.1}$$

Lemma 66 *For any four sets a, b, c, d, we have $(a, b) = (c, d)$ iff $a = c, b = d$.*

A triple (a, b, c) is a pair of form $((a, b), c)$. Clearly, $(a, b, c) = (a', b', c')$ iff $a = a', b = b', c = c'$.

Definition 129 *Given two sets A, B their* Cartesian product *is the set*

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

consisting of all ordered pairs (a, b), with the first coordinate *a an element in the* first factor *A, and the* second coordinate *b an element in the* second factor *B. A relation from A to B is a subset $R \subseteq A \times B$. One writes aRb for $(a, b) \in R$, and, if the relation is clear, more simply $a \sim b$. The inverse of a relation is the set $R^{-1} = \{(b, a) | aRb\}$. If $A = B$, one also speaks of a relation on A.*

Definition 130 *A* graph *$f : A \to B$* from *A* to *B is a triple (A, B, f) with f a relation from A to B; it is called*

(*i*) total *iff every $a \in A$ is the first factor of a pair in f;*
(*ii*) functional *iff $(a, b), (a, b') \in f$ implies $b = b'$; one writes $f(a)$ instead of the uniquely determined b, or also $f : a \mapsto b$.*
(*iii*) *a* function *or* map *iff it is a total functional graph. The set A is the* domain of *f, whereas B is the function's* codomain.

If $f : A \to B$ and $g : B \to C$ are two functions, their *composition* or *concatenation* function $g \circ f : A \to C$ is defined by $g \circ f(a) = g(f(a))$. The function $f : A \to A$ with $f(a) = a$ for all $a \in A$ is called the identity on A and is denoted by Id_A.

Exercise 83 *Verify that $g \circ f$ is indeed a function. If $h : C \to D$ is a third function, we have $(h \circ g) \circ f = h \circ (g \circ f)$ and therefore we write $h \circ g \circ f$. Show that $Id_B \circ f = f \circ Id_A = f$.*

Definition 131 *A function f is called*

(*i*) injective *if $f(a) = f(a')$ always implies $a = a'$;*
(*ii*) surjective, *iff for every $b \in B$, there is $a \in A$ such that $f(a) = b$;*
(*iii*) bijective, *iff f is injective and surjective.*

Lemma 67 *For a function $f : A \to B$, the following conditions are equivalent:*

(*i*) *There is a function $g : B \to A$ such that $g \circ f = Id_A$ and $f \circ g = Id_B$.*
(*ii*) *The function f is bijective.*

The g in this lemma is uniquely determined by f and is called the *inverse function of* f, it is denoted by f^{-1}. Bijections $f : A \to A$ are also called *permutations of* A. If there is a bijection $f : A \to B$ between two sets A, B, we say that *they have the same cardinality*, and write $card(A) = card(B)$. On $AllSet$, the cardinality relation is an equivalence relation, and one may define the *cardinality* $card(A)$ *of a set* A as the equivalence class $[A]$ under this relation. A finite set is one whose cardinality is that of a set of form $\{1, 2, 3, \ldots n\}$, with a natural number $0 \leqslant n$, where we take the empty set for $n = 0$.

Definition 132 *A binary relation $\leqslant$ on a set S is said to be*

(i) reflexive *iff $x \leqslant x$ for all $x \in S$;*
(ii) transitive *iff $x \leqslant y$ and $y \leqslant z$ implies $x \leqslant z$ for all $x, y, z \in S$;*
(iii) symmetric *iff $x \leqslant y$ implies $y \leqslant x$ for all $x, y \in S$;*
(iv) antisymmetric *iff $x \leqslant y$ and $x \neq y$ excludes $y \leqslant x$ for all $x, y \in S$;*
(v) total *iff $x \leqslant y$ or $y \leqslant x$ for all $x, y \in S$.*

Definition 133 *A binary relation $\leqslant$ on a set S is called an* equivalence relation *iff it is reflexive, transitive, and symmetric. In this case, the relation is usually denoted by "$\sim$" instead of "$\leqslant$".*

Lemma 68 *Let $\sim$ be an equivalence relation on S. Then the subsets $[s] = \{t | s \sim t\}$ are called* equivalence classes *of $\sim$. The set of equivalence classes is denoted by $S/\sim$. It defines a partition of S, i.e., it covers S, and for any two elements $s, t \in S$, either $[s] = [t]$ or $[s] \cap [t] = \varnothing$.*

Definition 134 *A binary relation $\leqslant$ on a set S is called a* partial ordering *iff it is reflexive, transitive, and antisymmetric. A partial ordering is called* linear *iff it is total. A linear ordering is called* well-ordered *iff every non-empty subset $T \subset S$ contains a minimal element.*

Lemma 69 *Let $\leqslant$ be a binary relation on a set S. Denoting $x < y$ iff $x \leqslant y$ and $x \neq y$, the following two statements are equivalent:*

(i) *The relation $\leqslant$ is a partial ordering.*
(ii) *The relation $\leqslant$ is reflexive, the relation $<$ is transitive, and for all $x, y \in S$, $x < y$ excludes $y < x$.*

If these equivalent properties hold, we have $x \leqslant y$ iff $x = y$ or else $x < y$. In particular, if we are given $<$ with the properties (ii)*, and if we* define *$x \leqslant y$ by the preceding condition, then the latter relation is a partial ordering.*

Proof. (i) $\Rightarrow$ (ii) Clearly $\leqslant$ is reflexive. If $x < y$ and $y < z$, then $x \leqslant z$. If we had $x = z$, then we were in the second statement of (ii), and it suffices to prove this one. But $x < y$ and $y < x$ implies $x = y$ by the asymmetry of $\leqslant$, a contradiction. (ii) $\Rightarrow$ (i) Transitivity: If $x \leqslant y$ and $y \leqslant z$, and either $x = y$ or $y = z$, we are done. Otherwise $x < y$ and $y < z$, whence $x < z$, therefore $x \leqslant z$. Asymmetry: If $x \leqslant y$ and $y \leqslant x$, then $x \neq y$ is excluded by (ii) whence the claim. The last statement is clear, QED.

Example 94 Suppose that $(I, <)$ is a linearly ordered "index set", and that we are given a family $((T_i, <_i, s_i))_I$ of linearly ordered sets $(T_i, <_i)$, each having a distinguished "constant" element s_i. Consider the set W of all nearly constant "words" $(t_i)_i \in \prod_i T_i$, i.e., $t_i = s_i$ except for a finite set of indexes. Then W has a canonical linear ordering, the *lexicographic ordering*, defined as follows. If we are given two different words, consider the smallest index where they differ. The smaller letter at that index defines the smaller word. Prove that this is a linear ordering.

In the usual lexicographic situation, the sets T_i are all an alphabet with its alphabetic linear ordering—as expressed, for example, by the $ASCII$ or $UNICODE$ encoding by natural numbers—, and the empty space as constant element. The indexes are natural, $i = 0, 1, 2, \ldots$.

Axiom 6 (Axiom of choice) *Let V be a partition of a set X. Then there is a subset $Y \subseteq X$ such that for every member M_i of the family V, the intersection $M_i \cap Y = \{m_i\}$ is a one-element set (a singleton).*

This is one of the many equivalent versions of the axiom of choice [561]. A particularly important variant is this:

Theorem 46 (Zermelo) *If the axiom of choice holds there is a well-ordered relation for every set. Conversely, if every set can be well ordered, the axiom of choice holds.*

For a proof, see [561, p.261].

Proposition 77 *Let $\leqslant$ be a partial ordering on a set S. Let $\preceq$ be the following binary relation on the set $Fin(S)$ of finite subsets of S. For $A, B \subset S$, let*

$$A \preceq B \begin{cases} A = B \text{ or} \\ A \neq B \text{ and for all } x \in A - B, \text{ there is } y \in B - A \text{ with } x < y. \end{cases} \tag{C.2}$$

Then $\preceq$ is a partial ordering. If $\leqslant$ is linear, so is $\preceq$.

Proof. We set $A \prec B$ as in Lemma 69. If $\leqslant$ is total, then for any $A \neq B$ in $Fin(S)$, if both disjoint difference sets $A - B$ and $B - A$ are non-empty, they contain different maximal elements, and we are done. Next, we verify statement (ii) about $\prec$. Suppose that we have $A \prec B$ and $B \prec A$ for $A \neq B$. Hence, we may suppose $A - B \neq \emptyset$. Take a maximal $x \in A - B$. There is $y \in B - A$ with $x < y$. Then there is $z \in A - B$ with $y < z$, but by transitivity of $<$, $x < z$, a contradiction to maximality of x. To show transitivity, take $A \neq B \neq C$ in $Fin(S)$. Since $A - C = \emptyset$ is trivial, suppose that x is a maximal element of $A - C$.

Suppose first that $x \in B$. Then $x \in B - C$, and there is $x < z$ with $z \in C - B$, take a maximal element z of this type. If $z \in A$, then $z \in A - B$, hence there is $z < w$, $w \in B - A$. If $w \in C$, $w \in C - A$, and $x < z < w$ which was required. Else, $w \notin C$ whence $w \in B - C$, and there is $u \in C - B$ with $w < u$, but $z < w < u$ contradicts maximality of z in $C - B$. Now if $z \notin A$, we have $z \in C - A$, and we are also done.

Suppose now that $x \notin B$, and therefore $x \in A - B$. There is $z \in B - A$ with $x < z$, take a maximal such element in $B - A$. If $z \in C$, then $z \in C - A$ and there is $w \in C - B$ with $z < w$. If $w \notin A$, then $w \in C - A$ and $x < z < w$ which was required. If $w \in A$ we have $w \in A - B$, and there is $u \in B - A$ with $u < w$. But then $z < w < u$ contradicts maximality of z in $B - A$, QED.

Definition 135 *For a finite partial ordering Rel on a set X, the* Hasse diagram *$Rel(X)$ is the relation whose pairs are the uniquely determined minimal set of generating relations of Rel.*

Proof of uniqueness. Suppose that two minimal generating sets A, B are different. Then any relation of $A - B$ points at a decomposition with a factor in $B - A$, and so forth, vice versa, such that we obtain an infinite chain of relations, contradicting the finiteness of X.

For a finite partial ordering Rel on X, the level function $lev : X \to \mathbb{N}$ is defined as follows. For minimal elements x in the Hasse diagram $Rel(X)$, put $lev(x) = 0$, for any element $x \in X$, set $pre(x) = \{y |\ y < x \text{ in } Rel(X)\}$. Then we put $lev(x) = 1 + Max(lev(y), y \in pre(x))$.

C.2.1 Universal Constructions

If $V = (M_i)_{i \in I}$ is a family of sets, the *product* set $\prod_I M_i$ is the set whose elements are the functions $f : I \to \bigcup V$ such that $f(i) \in M_i$ for all $i \in I$. Evidently, pairs, triples are special cases of such functions, but they are basic to the definition of functions, and therefore are treated separately. For each $j \in I$, we have the projection function $p_j : \prod_I M_i \to M_j$. Here is the *universal*[3] *property* of the product[4] set:

[3] This is a terminology from category theory, stating (in its most generic form) that a certain object in a category is final, but see Appendix G.

[4] This is a special case of a limit set, see Appendix G.2.1.

Lemma 70 *With the above notation, if $(f_i : X \to M_i)_i$ is a family of functions, then there is exactly one function $f : X \to \prod_I M_i$ such that $f_i = p_i \circ f$ for all $i \in I$, i.e., $f(x)(i) = f_i(x)$.*

For the given family V, the *coproduct* set $\coprod_I M_i$ is the union of the family $V' = (\{i\} \times M_i)_i$. For each index j, we have a function $\iota_j : M_j \to \coprod_I M_i$ defined by $\iota_j(m) = (j, m)$. The *universal property* of the coproduct[5] is this:

Lemma 71 *For every family $(f_i : M_i \to X)_i$ of functions, there is exactly one function $f : \coprod_I M_i \to X$ such that $f \circ \iota_i = f_i$, i.e., $f((i, m)) = f_i(m)$.*

Lemma 72 *If $f : A \to B$ is a function, then the* image *$Im(f) = \{b | b \in B,$ there exists $a \in A$ such that $b = f(a)\}$, together with the inclusion function $i : Im(f) \to B$ and the surjective function $f' : A \to Im(f) : a \mapsto f(a)$ has the following* universal property*: We have $f = i \circ f'$, and for every factorization $f = u \circ v$, there is a unique factorization $u = i \circ h$, and $f' = h \circ v$.*

Lemma 73 *Let A^B be a* powerset*, i.e., the set of functions $f : B \to A$. Then there is a natural bijection of* adjunction *$ad : C^{A \times B} \to (C^B)^A$, defined by $ad(g)(a)(b) = g(a, b)$.*

For a function $f : A \to B$ and a subset $C \subseteq B$, we define by $f^{-1}(C) = \{x | x \in A, f(x) \in C\}$ the *inverse image of C* under f; if $C = \{c\}$ is a singleton, we also write $f^{-1}(c)$ instead and call the set the *fiber of c*.

Lemma 74 *Let $2 = \{0, 1\}, 0 = \varnothing, 1 = \{0\}$, and let $Sub(X)$ be the set of subsets of X (also called the* powerset of X*). Then there is a natural bijection $\chi : Sub(X) \to 2^X$ defined by $\chi(Y)(z) = 0$ iff $z \in Y$, and $\chi(Y)(z) = 1$ iff $z \notin Y$. The function $\chi(Y)$ is called the* characteristic function of Y*. The inverse function maps $c : X \to 2$ to $Y_c = c^{-1}(0)$.*

C.2.2 Graphs and Quivers

Definition 136 *For a set X, denote by $P_2(X)$ the set of subsets of cardinality one or two in X, i.e., the set of singletons and unordered pairs in X. Then a* multigraph *is a triple $(L, V, G : L \to P_2(V))$. The elements of L are called the* lines*, the elements of V are called the* vertexes *of the multigraph. Often, we identify the multigraph with its map G. If G is injective, the multigraph is called a* graph*. A multigraph is finite, iff all involved sets are finite.*

Definition 137 *A* quiver *is a pair $G = (\mathrm{head}, \mathrm{tail} : A \rightrightarrows V)$ of set maps. The elements of V are called* vertexes*, the elements of A are called* arrows*. If every pair of vertexes is head and tail of at most one arrow, the quiver is called a* directed graph*.*

A path *p in G is either a vertex $p = v$ (a 'lazy' path) or else a sequence $p = (a_0, a_1, \ldots a_n)$ of arrows with $\mathrm{head}(a_i) = \mathrm{tail}(a_{i+1})$ for each index $i = 0, 1, \ldots n-1$. The length $l(p)$ of a path is 0 for a lazy path, and n for a general path. A* closed path *is a path with $\mathrm{head}(a_n) = \mathrm{tail}(a_0)$. A lazy path is also a closed path of length 0. A closed path of length 1 is also called a* loop*. A* cycle *is an equivalence class of closed paths which differ from each other by their start/end point.*

We also use the symbols $\mathrm{head}(p), \mathrm{tail}(p)$ for the head, or tail, respectively, of the last, or first arrow of p, respectively (or just the unique vertex v for the lazy path).

Example 95 For a set V, a *complete* quiver is a quiver $G = (head, tail : A \rightrightarrows V)$ such that the map

$$f : A \to V^2 : a \mapsto (\mathrm{tail}(a), \mathrm{head}(a))$$

is a bijection.

For two paths p, q with $\mathrm{head}(q) = \mathrm{tail}(p)$, we have a composed path $p.q$ which is the evident juxtaposition of arrow sequences or the respectively other path if one path is lazy. As with functions, the composition of paths is associative if it is defined.

[5] This is a special case of a colimit set, see Appendix G.2.1.

C.2.3 Monoids

Definition 138 *A* semigroup *is a couple (M, μ) consisting of a set M and an associative binary operation $\mu : M \times M \to M : (m, n) \mapsto \mu(m, n) = m.n$. The semigroup M is a* monoid *if there exists a* neutral element *e, i.e., $e.m = m.e = m$ for all $m \in M$. Since e is uniquely determined by this property, it is called* the *neutral element. For a given subset $S \subset M$ of a semigroup, the semigroup $\langle S \rangle$ generated by S is the smallest sub-semigroup of M containing S. If M is a monoid, the submonoid with neutral element e generated by S is denoted by $\langle S \rangle_e$. If $m.n = n.m$ for all $m, n \in M$, the semigroup is called* commutative. *An element $x \in M$ such that there is y with $x.y = e$ is called invertible, its set $M^\star$ is a submonoid. A monoid (M, μ) is* finite, *iff the underlying set M is so.*

Given two monoids M, N a monoid homomorphism *is a set map $f : M \to N$ such that $f(e_M) = e_N$ and $f(m.m') = f(m).f(m')$ for all $m, m' \in M$. A* monoid isomorphism *is a bijective monoid homomorphism, its inverse set map is then automatically a monoid isomorphism.*

Exercise 84 *Show that the map $N \mapsto N_e = N \cup \{e\}$ defines a projection from the set of sub-semigroups of M onto the set of sub-monoids of M with $N_e = N$ iff N is a submonoid and $N_e = N'_e$ iff $N = N'$ for sub-semigroups N, N' which are not sub-monoids.*

Example 96 For a set[6] *Alphabet*, there is a monoid $FM(Alphabet)$ and an injection

$$i : Alphabet \to FM(Alphabet)$$

such that for any set map $f : Alphabet \to M$ into a monoid M, there is exactly one monoid homomorphism $FM(f) : FM(Alphabet) \to M$ such that $FM(f) \circ i = f$. One calls the monoid $FM(Alphabet)$ the *free monoid over Alphabet* and the above property is the universal property that characterizes the free monoid up to isomorphism. The free monoid consists of all "word" expressions $b_1 \dots b_k$ for $b_i \in Alphabet, k > 0$, and the empty word (). The product is defined by juxtaposition of words, () is neutral. The free monoid is also called "word monoid over the alphabet *Alphabet*".

Consider the equivalence relation on $FM(Alphabet)$ with $b_1 \dots b_k \sim b_{\pi(1)} \dots b_{\pi(k)}$ for any permutation π, then the quotient space $FCM(Alphabet) = FM(Alphabet)/\sim$ with the induced multiplication is a commutative monoid and is called the *free commutative monoid over Alphabet*.

Definition 139 *In a semigroup M, an* idempotent *is an element x such that $x = x.x = x^2$. We denote $Idempot(M) = \{x |\ x \text{ is idempotent in } M\}$. For a subset $X \subset M$, we define its* radical *by*

$$\sqrt{X} = \{y | \exists n, 0 < n, \text{ such that } y^n \in X\}.$$

In particular, given an idempotent x, the radical $\sqrt{x}$ of x is defined by

$$\sqrt{x} = \{y | \exists n, 0 < n, \text{ such that } x = y^n\}.$$

Since for every $y \in \sqrt{x}$, $\langle y \rangle \subset \sqrt{x}$, we also have $\sqrt{x} = \bigcup_{y \in \sqrt{x}} \langle y \rangle$.

Example 97 For example, the neutral element in a monoid is idempotent.

Example 98 If Q is a left module[7] over a ring R, the set of idempotents $Idempot(End(Q))$ of the linear endomorphism ring of Q is in bijection with the set $Dir(Q)$ of direct decompositions $Q = U \oplus V$ of Q. In fact, the bijection is set up by

$$Idempot(End(Q)) \xrightarrow{\sim} Dir(Q) : x \mapsto (Im(x), Im(Id_Q - x))$$

[6] E.g. $Alphabet = ASCII, UNICODE$.
[7] See Appendix E.

whose inverse defines the projection $p : Q \to U$ onto the factor U of $Q = U \oplus V$. The image of an idempotent x is a direct decomposition by the equation $Id_Q = x + (Id_Q - x)$ which yields $Q = Im(x) \oplus Im(Id_Q - x)$ with $x = Id_{Im(x)} \oplus 0_{Im(Id_Q - x)}$. Conversely, the projection $x = pr_U$ yields $U = Im(x), V = Im(Id_Q - x)$. So the above map is a bijection.

Lemma 75 *The idempotents of the monoid $Q@Q$ of affine endomorphisms*[8] *are the elements $e^q.y$ such that $y \in Idempot(End(Q))$ and $q \in Im(Id_Q - y) = Ker(y)$.*

Example 99 For example, the idempotents of $\mathbb{Z}_{12}$ are $0, 1, 4, 9$, and $Ker(0) = \mathbb{Z}_{12}, Ker(1) = 0, Ker(4) = 3.\mathbb{Z}_{12}, Ker(9) = 4.\mathbb{Z}_{12}$, so we have a total of 20 affine idempotents here.

Lemma 76 *With the previous notation, for two idempotent elements x, y of a semigroup M, $\sqrt{x} \cap \sqrt{y} \neq \varnothing$ iff $x = y$. Hence we have a partition of radicals in M, viz., $\sqrt{Idempot(M)} = \coprod_{x \in Idempot(M)} \sqrt{x}$, the* idempotent components[9] *of M.*

Clearly, if $f : M \xrightarrow{\sim} M$ is an automorphism of the semigroup M, we have $f(\sqrt{x}) = \sqrt{f(x)}$. In particular, if $M = Q@Q$ for a module Q, and if $y \in End(Q)$ is an idempotent linear endomorphism of Q, the conjugation by a translation $e^q, q \in Q$ gives the formula

$$e^q \sqrt{y} e^{-q} = \sqrt{e^{(1-y)q}.y} \tag{C.3}$$

which means that the translation exponent $(1 - y)q$ is a general element of $Ker(y)$. Therefore:

Proposition 78 *For a module Q, the orbits of idempotent components under the conjugation action of the translation group e^Q are the sets of idempotent components associated with the linear idempotent endomorphisms of Q.*

Lemma 77 *The canonical surjective linear factor projection $p : Q@Q \to End(Q)$ is compatible with the idempotent partitions, and the fiber of every idempotent component in $End(Q)$ is the orbit of the translation group action under conjugation (see Proposition 78).*

Proposition 79 *If in the above example the module Q is finite, the respective idempotent components define partitions*

$$\sqrt{Idempot(End(Q))} = End(Q)$$

and

$$\sqrt{Idempot(Q@Q)} = Q@Q.$$

Proof. Let $y \in End(Q)$. By Fitting's lemma (Appendix 88), there is a positive power $z = y^n$ and a direct decomposition $Q = U \oplus V$ such that $z|U \in GL(U), Ker(z) = V$. So by finiteness of U, there is a positive power $(z|U)^m = Id_U$. Therefore, $z^m = y^{nm}$ is an idempotent. As to the affine case, if $y = e^t.y_0 \in Q@Q$, we know by the preceding that a power $y_0^m = t$ is idempotent. So WLOG., we may suppose that y_0 is idempotent. Set $e^{t_k}.y_0 = (e^t.y_0)^k = y^k$; the positive powers of y must have recurrent values, so take $y^u = y^{u+k}$, for positive k. Then $e^{t_u}.y_0 = e^{t_u}.y_0.e^{t_k}.y_0 = e^{t_u + y_0(t_k)}.y_0$, i.e., $t_k \in Ker(y_0)$, whence y^k is an idempotent by Lemma 75, QED.

Lemma 78 *For the subgroup*[10] *$M^\star$ of invertible elements of the monoid M, we have the intersection formula*

$$M^\star \cap \sqrt{Idempot(M)} = \sqrt{1_M}.$$

[8] See Appendix E.
[9] Terminology of Noll, [802].
[10] See section C.3.

Proof. Clearly, $\sqrt{1_M} \subset M^\star$. On the other hand, if $x \in \sqrt{t} \subset M^\star$, we have $x^k = t = t^2 = x^{2k}$, whence $1_M = x^k = t$.

In the special case of a total partition $M = \sqrt{Idempot(M)}$, we get

$$M^\star = \sqrt{1_M}.$$

This is the case for an affine endomorphism monoid $Q@Q$ of a finite module Q.

C.3 Groups

Definition 140 *A monoid (M, μ) with the neutral element e such that every element m has a left inverse n, i.e., $n.m = e$, is called a* group. *A left inverse of m is also a right inverse, $m.n = e$, and the inverse is uniquely determined by m, it is denoted by m^{-1}. A subgroup of a group M is a submonoid N which is a group. A commutative group is also called* abelian. *If $n \in \mathbb{N}$, then we write m^n for the n-fold product $m.m.\ldots.m$; if n is a negative integer, then we write $m^n = (m^{-1})^{|n|}$, we also set $m^0 = e$. Evidently $m^n.m^l = m^{n+l}$.*

If a group is abelian, the product is usually written additively, i.e., $m+n$ instead of $m.n$, further $m-n$ instead of $m.n^{-1}$, and the neutral element is noted by 0 instead of e.

Exercise 85 *Observe that the neutral element e of a group is the only idempotent element $x.x = x$. Therefore, a subgroup has necessarily the same neutral element as the supergroup.*

C.3.1 Homomorphisms of Groups

If (G, γ), (H, η) are two groups, a set map $f : G \to H$ is called a *group homomorphism* iff $f(g.g') = f(g).f(g')$ (products in the respective groups). We have $f(e_G) = e_H$ and $f(x^{-1}) = f(x)^{-1}$. Clearly, the set-theoretic composition of two group homomorphisms is also a group homomorphism. The set of group homomorphisms $f : G \to H$ is denoted by $Hom(G, H)$. If f is a bijection, its inverse is also a group homomorphism; f is then called a *group isomorphism*. G, H are said to be *isomorphic* iff there is an isomorphism between them. Clearly, group isomorphisms are an equivalence relation, whence the term "isomorphism classes". The explicit description of isomorphism classes is the main task of group theory. For finite groups, this is essentially solved[11]. For finite commutative groups, the classification is described in Appendix C.3.4.

The set $Aut(G)$ of isomorphisms of a group G onto itself (the *automorphisms* of G) is a group under the composition of group homomorphisms, the identity Id_G being the neutral element of $Aut(G)$. The group $\mathfrak{S}_M$ of permutations of a set M is called the *symmetric group of M*.

Example 100 A homomorphism $\mu : G \to \mathfrak{S}_M$ is called a *left action of G on M*. Let G^{opp} denote the *opposite group to G* where products are interchanged: $g.^{opp}h = h.g$. Then a right action is a homomorphism $\mu : G^{opp} \to \mathfrak{S}_M$. A left action is equivalent to a map $\mu : G \times M \to M$ (same notation for μ) with $\mu((e, m) = m, \mu(g(\mu(h, m)) = \mu(g.h, m)$, all $g, h \in G, m, n \in M$. If the action is clear, we write $g.m$ instead of $\mu(g, m)$. We usually mean *left* actions when we speak of actions.

A group action μ is called *effective* iff it is an injective homomorphism. For an element $m \in M$, the group $G_m = \{g \in G, g.m = m\}$ is called the *stabilizer* or *isotropy* or *fixpoint group of m*. If all stabilizers are trivial, the action is called *(fixpoint) free* or *faithful*.

For a group action μ, an *orbit* is a set of form $G.m = \{g.m | g \in G\}$. The *orbit space* $G\backslash M$ defines a partition of M, with the canonical map $\pi : M \to G\backslash M : m \mapsto G.m$. The action is *transitive* iff the orbit space is a singleton. Clearly

$$card(G.m).card(G_m) = card(G).$$

If $\mu_1 : G \times M \to M, \mu_1 : G \times N \to N$ are two group actions of group G, and if $h : M \to N$ is a set map, we say that h is *equivariant* iff it commutes with these group actions, i.e., if for all $m \in M, g \in G$, we have $h(\mu_1(g, m)) = \mu_2(g, h(m))$.

[11] The classification of all finite simple groups is one of the main results in group theory of the 20^{th} century, see [381].

Example 101 For a set *Alphabet*, there is a group $FG(Alphabet)$ and an injection $i : Alphabet \to FG(Alphabet)$ such that for any set map $f : Alphabet \to G$ into a group G, there is exactly one group homomorphism $FG(f) : FG(Alphabet) \to G$ such that $FG(f) \circ i = f$. The group $FG(Alphabet)$ is called the *free group over Alphabet* and the above property is the universal property that characterizes the free group up to isomorphism. The free group consists of all "word" expressions $b_1^{n_1} \dots b_k^{n_k}$, where $b_i \in Alphabet, n_i \in \mathbb{Z}$ which are reduced, i.e., $b_i \neq b_{i+1}, n_i \neq 0$, and the empty word (). The product is defined by juxtaposition and cancelling of powers of adjacent letters.

Lemma 79 *For a group homomorphism $f : G \to H$, $Im(f)$ is a subgroup of H. For a subgroup $I \subseteq H$, the inverse image $f^{-1}(I)$ is a subgroup of G. The inverse image of the trivial group (the singleton consisting of the neutral element e_H) in H is called the* kernel of f *and is denoted by $Ker(f)$*

Example 102 For every element $g \in G$ of a group, we have a special group automorphism

$$Int_g : G \to G : h \mapsto g.h.g^{-1}$$

the *conjugation with g*. This yields a group homomorphism $Int : G \to Aut(G)$.

Proposition 80 *For a group G, a subgroup H and an element $g \in G$, we write $gH = \{g.h | h \in H\}, Hg = \{h.g | h \in H\}$ for the* left and right cosets *of H. The set of left, right cosets of G is denoted by G/H, $H\backslash G$, respectively. Denoting $(G : H) = card(G/H)$, we have the Lagrange equation*[12]*:*

$$card(G) = card(H).(G : H)$$

and in particular, if G is finite, any subgroup cardinality divides the order $card(G)$ of G.

Proposition 81 *If H is a subgroup of G, the following statements are equivalent:*

(*i*) *For all $g \in G$, $gH = Hg$.*
(*ii*) *There is a group homomorphism $f : G \to K$ with $Ker(f) = H$.*

If H has the properties of Proposition 81, it is called a *normal subgroup of G*, in symbols: $H \lhd G$. A *simple* group is one which has no normal subgroup except the trivial and the full subgroup. The group homomorphism in Proposition 81 is constructed by the *quotient group G/H* structure on G/H, where the product is defined by $gH.g'H = g.g'H$. The *canonical* homomorphism f is defined by $K = G/H$, and $f(g) = gH$.

Proposition 82 *Let $H \lhd G$ be a normal subgroup, $\pi : G \to G/H$ the canonical homomorphism, and K any group. Then*

$$Hom(G/H, K) \xrightarrow{\sim} \{f \in Hom(G, K) | H \subseteq Ker(f)\} : t \mapsto t \circ \pi$$

is a canonical bijection. If $H = Ker(f)$, then the corresponding morphism $G/H \to K$ is an isomorphism onto $Im(f)$.

Exercise 86 *If $f : G \to H$ is an isomorphism of groups, then the map*

$$Int_f : Aut(G) \to Aut(H) : t \mapsto f.t.f^{-1}$$

is an isomorphism of groups.

Here are the basic isomorphism theorems:

Theorem 47 (First isomorphism theorem) *Let G, H, N be groups with $N \lhd G, H \subseteq G$. Then:*

[12] Define $card(A).card(B) = card(A \times B)$.

(*i*) *The product set* $H.N$ *is a subgroup.*
(*ii*) *We have* $N \lhd H.N$ *and* $N \cap H \lhd H$.
(*iii*) *We have an isomorphism* $H/N \cap H \xrightarrow{\sim} H.N/N : h(N \cap H) \mapsto hN$.

Theorem 48 (Second isomorphism theorem) *Let* $N \lhd G$ *be a normal subgroup.*

(*i*) *The normal subgroups* $\bar{M} \lhd G/N$ *are in bijection with the subgroups* M *with* $N \lhd M \lhd G$, *and*
(*ii*) *we have an isomorphism* $G/M \xrightarrow{\sim} (G/N)/(M/N) : gM \mapsto (gN).M/N$.

C.3.2 Direct, Semi-direct, and Wreath Products

Given a family $(G_i)_{i\in I}$ of groups, the set-theoretic product $\prod_I G_i$ becomes a *(direct) product group* via the coordinate-wise product, i.e., $(g_i).(h_i) = (g_i.h_i)$.

Given two groups H, N and a group homomorphism $\phi : H \to Aut(N)$, the *semidirect product* $N \rtimes_\phi H$ is the group structure on the set $N \times H$ defined by

$$(n, g).(m, h) = (n.{}^g m, g.h)$$

where ${}^g m = \phi(g)(m)$. We have two group injections $i : N \to N \rtimes_\phi H : n \mapsto (n, e_H)$ and $j : H \to N \rtimes_\phi H : h \mapsto (e_N, h)$, and $i(N) \lhd N \rtimes_\phi H$. Further, $i(N) \cap j(H) = e$, $i(N).j(H) = N \rtimes_\phi H$, and the conjugation action identifies to the group homomorphism ϕ, i.e., ${}^g m = Int_g(m)$. We therefore have the short exact sequence (i is injective, π is surjective, and $Ker(\pi) = Im(i)$):

$$e \longrightarrow N \xrightarrow{i} N \rtimes_\phi H \underset{\pi}{\overset{j}{\rightleftarrows}} H \longrightarrow e$$

which is split, i.e., $\pi \circ j = Id_H$. Conversely, any such split sequence

$$e \longrightarrow N \xrightarrow{i} G \underset{\pi}{\overset{j}{\rightleftarrows}} H \longrightarrow e$$

identifies the middle group G to $N \rtimes_\phi H$.

Example 103 For two groups G, H, we have the direct product group $G^H = \prod_H G$ and an action $\phi : H \to Aut(G^H)$ via $\phi(h)(f)_k = f_{k.h}, k \in H$. Then the wreath product is $G \wr H = G^H \rtimes_\phi H$.

C.3.3 Sylow Theorems on *p*-groups

Proposition 83 (Sylow's proposition) *If* G *is a finite group of order* n, p *is prime, and* $p^k | n$, *then there is a subgroup* $H \subseteq G$ *with order* p^k.

A finite group G has the order $card(G) = p^k$ for a prime p iff all its elements have a power of p as their orders. In this case, G is called a *p-group*. A maximal p-subgroup in a group G is called a *p-Sylow group* in G.

Theorem 49 (Sylow's Theorem) *Let* G *be a finite group,* p *a prime. Then:*

(*i*) *The p-Sylow groups in* G *are the p-subgroups* S *with* $p \nmid (G : S)$.
(*ii*) *Any two p-Sylow groups are conjugate to each other.*
(*iii*) *Let* σ_p *be the cardinality of the set of p-Sylow groups in* G. *Then we have* $\sigma_p | card(G)$ *and* $\sigma_p \equiv 1 \pmod{p}$.

See [41] for a proof.

C.3.4 Classification of Groups

C.3.4.1 Classification of Cyclic Groups

If $S \subset G$ is a subset of a group G, the smallest subgroup in G containing S is denoted by $\langle S \rangle$ and is called the *group generated by* S. A *finitely generated* group is one that admits a finite set of generators. A cyclic group is one that is generated by one element $G = \langle s \rangle$. For such a group, the group homomorphism $s^? : \mathbb{Z} \to G : n \mapsto s^n$ is a surjection with kernel $Ker(s^?) = O(s).\mathbb{Z}$, where $s^{O(s)} = e$ is the smallest positive power of s which yields e, the *order of* s, or $O(s) = 0$ if no positive power of s vanishes, in which case the order of s is said to be infinite. This means that $\langle s \rangle \tilde{\to} \mathbb{Z}/O(s).\mathbb{Z}$, and we have classified all cyclic groups: They are isomorphic to the quotient groups $\mathbb{Z}_n = \mathbb{Z}/n.\mathbb{Z}$ with $card(\mathbb{Z}_n) = n$ for positive n, and ∞ for $\mathbb{Z}_0 = \mathbb{Z}$.

Among these groups, the groups $\mathbb{Z}_p$ of prime order p are *simple*. For abelian groups, simplicity means being cyclic of prime order.

The number of generators of a finite cyclic group $\mathbb{Z}_n$ is the number of numbers $0 < t < n$ prime to n, i.e., the *Euler function* $\phi(n) = card(\mathbb{Z}_n^\times)$, where $\mathbb{Z}_n^\times$ denotes these numbers modulo n, this is in fact the group of invertible elements of the ring $\mathbb{Z}_n$, see Appendix D.1.

Proposition 84 *If $h = r.s$ is a factorization by natural numbers $1 < r, s$, and $(r, s) = 1$, then $\phi(h) = \phi(r).\phi(s)$. If $h = p^k$ is a positive power of a prime number q, then $\phi(p^k) = q^{k-1}(q-1)$. Therefore, if $h = \prod_i q_i^{k_i}$ is the prime decomposition of h, then*

$$\phi(h) = h.\prod_i (1 - 1/q_i).$$

C.3.4.2 Classification of Finitely Generated Abelian Groups

Let p be an prime integer ($p = 2, 3, 5, 7, 11, \ldots$). Take a weakly increasing sequence $u. = u_1 \leqslant u_2 \leqslant \ldots u_w$ of positive integers. We set

$$S(p, u.) = \mathbb{Z}_{p^{u_1}} \times \mathbb{Z}_{p^{u_2}} \times \ldots \mathbb{Z}_{p^{u_w}}.$$

If we are given an increasing sequence $p. = (p_1 < p_2 < \ldots p_t)$ of primes and for each such p_i a sequence $u.^i$, i.e., a sequence of sequences $u.^\cdot$. Then we set

$$T(p., u.^\cdot) = \prod_i S(p_i, u.^i). \tag{C.4}$$

Theorem 50 *For every finitely generated abelian group G, there is a natural number f and a system of primes $p.$ and positive, weakly increasing sequences $u.^\cdot$ such that*

$$G \tilde{\to} \mathbb{Z}^f \times T(p., u.^\cdot).$$

*All the numbers f (the torsion-free rank of G), and $p., u.^\cdot$ are uniquely determined. The length of the sequence $u.^i$ is called the p_i-*rank of G. *The image $T(G)$ of the factor $T(p., u.^\cdot)$ in G is the* torsion group *of G. The subgroups in G corresponding to $S(p_i, u.^i)$ are called the p_i-*Sylow groups. *They are the subgroups of all elements with an order equal to a power of p_i.*

For a proof, see [1078, Vol.II].

Corollary 31 *If G is a finite abelian group, and if $m|card(G)$, then there is a subgroup H of G with $card(H) = m$.*

C.3.5 General Affine Groups

For a commutative ring R, we denote by $GL_n(R)$ the group $\mathbb{M}_{n,n}(R)^\times$ of invertible $n \times n$-matrices over R, see Appendix D.1; we also denote $GL(n,p) = GL_n(\mathbb{Z}_p)$ and $\overrightarrow{GL}(n,p) = e^{\mathbb{Z}_p} \cdot GL(n,p)$, the affine automorphism group of $\mathbb{Z}_p^n$.

Theorem 51 (Minkowski) *Suppose that G is a finite subgroup of $GL_n(\mathbb{Z})$, and let $q \in \mathbb{N}$. Consider the canonical projection homomorphism*

$$\Phi : G \to GL_n(\mathbb{Z}_q).$$

If $3 \leqslant q$, then $Ker(\Phi)$ is trivial. If $q = 2$, then there is $U \in GL_n(\mathbb{Z})$ with

$$UNU^{-1} = Diag(\epsilon_1, \dots \epsilon_n) \text{ diagonal matrix,}$$

where $\epsilon_i = \pm 1$ for all $N \in Ker(\Phi)$. In particular, $card(Ker(\Phi))|w^n$.

For a proof, see [536, Satz 5.1].

For a prime number p and a power n, we have the following cardinalities:

$$card(GL(n,p)) = (p^n - 1)(p^{n-1} - 1) \dots (p-1)p^{n(n-1)/2}$$

and

$$card(\overrightarrow{GL}(n,p)) = p^n card(GL(n,p)).$$

C.3.6 Permutation Groups

Proposition 85 (Cayley) *Every group G is isomorphic to a subgroup of permutations.*

Proof. In fact, the left regular representation map $l_? : G \to \mathfrak{S}_G$ with $l_?(g) = l_g : h \mapsto g \cdot h$ is such an embedding, QED.

A permutation group $G \subseteq \mathfrak{S}_P$ by definition acts on the underlying set P. Suppose that P is finite. For $g \in G$, the orbits of $\langle g \rangle$ are called the cycles of g. Then these finite sets are arranged as sequences $C = (x, g.x, g^2.x, \dots g^{k(x)}.x)$ of pairwise different elements, i.e., $g^{k(x)+1}$ is the generator of the stabilizer of x. The permutation g can be represented as sequence $(C_1, C_2, \dots C_r)$ of orbits in cycle representation.

Definition 141 *If $G \subseteq \mathfrak{S}_P$ is a permutation group on a finite set P of cardinality p, the* cycle index of an element $g \in G$ *is the polynomial*

$$X_\cdot^{cyc(g)} = X_1^{c_1} \cdot X_2^{c_2} \cdot \dots X_p^{c_p}$$

with the cycle type of g *$cyc(g) = (c_1, c_2, \dots c_p)$, and $c_i = card(\{C = \text{ cycle of } g, card(g) = i\})$.*

Here is Fripertinger's cycle index formula [343]

$$Z(\overrightarrow{GL}(\mathbb{Z}_{12}^2)(1 + x, 1 + x^2, \dots 1 + x^{144})$$

for orbits of zero-addressed local compositions in $\mathbb{Z}_{12}^2$:

$x^{144} + x^{143} + 5x^{142} + 26x^{141} + 216x^{140} + 2\,024x^{139} + 27\,806x^{138} + 417\,209x^{137} + 6\,345\,735x^{136} + 90\,590\,713x^{135} +$
$1\,190\,322\,956x^{134} + 14\,303\,835\,837x^{133} + 157\,430\,569\,051x^{132} + 1\,592\,645\,620\,686x^{131} + 14\,873\,235\,105\,552x^{130} +$
$128\,762\,751\,824\,308x^{129} + 1\,037\,532\,923\,086\,353x^{128} + 7\,809\,413\,514\,931\,644x^{127} + 55\,089\,365\,597\,956\,206x^{126} +$
$365\,290\,003\,947\,963\,446x^{125} + 2\,282\,919\,558\,918\,081\,919x^{124} + 13\,479\,601\,808\,118\,798\,229x^{123} +$
$75\,361\,590\,622\,423\,713\,249x^{122} + 399\,738\,890\,367\,674\,230\,448x^{121} + 2\,015\,334\,387\,723\,540\,077\,262x^{120} +$
$9\,673\,558\,570\,858\,327\,142\,094x^{119} + 44\,275\,002\,111\,552\,677\,715\,575x^{118} + 193\,497\,799\,414\,541\,699\,555\,587x^{117} +$
$808\,543\,433\,959\,017\,353\,438\,195x^{116} + 3\,234\,171\,338\,137\,153\,259\,094\,292x^{115} +$
$12\,397\,650\,890\,304\,440\,505\,241\,198x^{114} + 45\,591\,347\,244\,850\,943\,472\,027\,532x^{113} +$

$$160\,994\,412\,344\,908\,368\,725\,437\,163x^{112} + 546\,405\,205\,018\,625\,434\,948\,486\,100x^{111} +$$
$$1\,783\,852\,127\,215\,514\,388\,216\,575\,524x^{110} + 5\,606\,392\,061\,138\,587\,678\,507\,139\,578x^{109} +$$
$$16\,974\,908\,597\,922\,176\,404\,758\,662\,419x^{108} + 49\,548\,380\,452\,249\,950\,392\,015\,617\,673x^{107} +$$
$$139\,517\,805\,378\,058\,810\,895\,892\,716\,876x^{106} + 379\,202\,235\,047\,824\,659\,955\,968\,634\,895x^{105} +$$
$$995\,405\,857\,334\,028\,240\,446\,249\,995\,969x^{104} + 2\,524\,931\,913\,311\,378\,421\,460\,541\,875\,013x^{103} +$$
$$6\,192\,094\,899\,403\,308\,142\,319\,324\,646\,830x^{102} + 14\,688\,225\,057\,065\,816\,000\,841\,247\,153\,422x^{101} +$$
$$33\,716\,152\,882\,551\,682\,431\,054\,950\,635\,828x^{100} + 74\,924\,784\,036\,765\,597\,482\,162\,224\,697\,378x^{99} +$$
$$161\,251\,165\,409\,134\,463\,248\,992\,354\,275\,261x^{98} + 336\,225\,833\,888\,858\,733\,322\,982\,932\,904\,265x^{97} +$$
$$679\,456\,372\,086\,288\,422\,448\,712\,466\,252\,503x^{96} + 1\,331\,179\,830\,182\,151\,403\,666\,404\,596\,530\,852x^{95} +$$
$$2\,529\,241\,676\,111\,626\,447\,928\,668\,220\,456\,264x^{94} + 4\,661\,739\,558\,127\,027\,290\,220\,867\,616\,981\,880x^{93} +$$
$$8\,337\,341\,899\,567\,786\,249\,391\,103\,289\,453\,916x^{92} + 14\,472\,367\,067\,576\,451\,752\,984\,797\,361\,008\,304x^{91} +$$
$$24\,388\,618\,572\,337\,747\,341\,932\,969\,998\,362\,288x^{90} + 39\,908\,648\,567\,034\,355\,259\,311\,114\,115\,744\,392x^{89} +$$
$$63\,426\,245\,036\,529\,210\,051\,949\,169\,850\,308\,102x^{88} + 97\,921\,220\,397\,909\,924\,969\,018\,620\,386\,852\,352x^{87} +$$
$$146\,881\,830\,585\,458\,073\,270\,850\,321\,720\,445\,928x^{86} + 214\,098\,939\,483\,879\,341\,610\,433\,150\,629\,060\,274x^{85} +$$
$$303\,306\,830\,919\,747\,863\,651\,620\,555\,026\,700\,930x^{84} + 417\,668\,422\,888\,061\,171\,460\,770\,548\,484\,103\,836x^{83} +$$
$$559\,136\,759\,653\,084\,522\,330\,064\,385\,877\,590\,780x^{82} + 727\,765\,306\,194\,069\,123\,565\,702\,210\,626\,823\,392x^{81} +$$
$$921\,077\,965\,629\,957\,077\,012\,552\,741\,715\,036\,692x^{80} + 1\,133\,634\,419\,214\,796\,834\,928\,853\,170\,296\,724\,314x^{79} +$$
$$1\,356\,926\,047\,220\,511\,677\,349\,073\,201\,120\,481\,570x^{78} + 1\,579\,704\,950\,475\,555\,411\,914\,967\,237\,903\,930\,342x^{77} +$$
$$1\,788\,783\,546\,844\,376\,088\,722\,000\,995\,922\,467\,990x^{76} + 1\,970\,254\,341\,437\,213\,013\,502\,048\,964\,983\,877\,090x^{75} +$$
$$2\,110\,986\,794\,386\,177\,596\,749\,436\,553\,816\,924\,660x^{74} + 2\,200\,183\,419\,494\,435\,885\,449\,671\,402\,432\,366\,956x^{73} +$$
$$2\,230\,741\,522\,540\,743\,033\,415\,296\,821\,609\,381\,912x^{72} + 2\,200\,183\,419\,494\,435\,885\,449\,671\,402\,432\,366\,956x^{71} +$$
$$2\,110\,986\,794\,386\,177\,596\,749\,436\,553\,816\,924\,660x^{70} + 1\,970\,254\,341\,437\,213\,013\,502\,048\,964\,983\,877\,090x^{69} +$$
$$1\,788\,783\,546\,844\,376\,088\,722\,000\,995\,922\,467\,990x^{68} + 1\,579\,704\,950\,475\,555\,411\,914\,967\,237\,903\,930\,342x^{67} +$$
$$1\,356\,926\,047\,220\,511\,677\,349\,073\,201\,120\,481\,570x^{66} + 1\,133\,634\,419\,214\,796\,834\,928\,853\,170\,296\,724\,314x^{65} +$$
$$921\,077\,965\,629\,957\,077\,012\,552\,741\,715\,036\,692x^{64} + 727\,765\,306\,194\,069\,123\,565\,702\,210\,626\,823\,392x^{63} +$$
$$559\,136\,759\,653\,084\,522\,330\,064\,385\,877\,590\,780x^{62} + 417\,668\,422\,888\,061\,171\,460\,770\,548\,484\,103\,836x^{61} +$$
$$303\,306\,830\,919\,747\,863\,651\,620\,555\,026\,700\,930x^{60} + 214\,098\,939\,483\,879\,341\,610\,433\,150\,629\,060\,274x^{59} +$$
$$146\,881\,830\,585\,458\,073\,270\,850\,321\,720\,445\,928x^{58} + 97\,921\,220\,397\,909\,924\,969\,018\,620\,386\,852\,352x^{57} +$$
$$63\,426\,245\,036\,529\,210\,051\,949\,169\,850\,308\,102x^{56} + 39\,908\,648\,567\,034\,355\,259\,311\,114\,115\,744\,392x^{55} +$$
$$24\,388\,618\,572\,337\,747\,341\,932\,969\,998\,362\,288x^{54} + 14\,472\,367\,067\,576\,451\,752\,984\,797\,361\,008\,304x^{53} +$$
$$8\,337\,341\,899\,567\,786\,249\,391\,103\,289\,453\,916x^{52} + 4\,661\,739\,558\,127\,027\,290\,220\,867\,616\,981\,880x^{51} +$$
$$2\,529\,241\,676\,111\,626\,447\,928\,668\,220\,456\,264x^{50} + 1\,331\,179\,830\,182\,151\,403\,666\,404\,596\,530\,852x^{49} +$$
$$679\,456\,372\,086\,288\,422\,448\,712\,466\,252\,503x^{48} + 336\,225\,833\,888\,858\,733\,322\,982\,932\,904\,265x^{47} +$$
$$161\,251\,165\,409\,134\,463\,248\,992\,354\,275\,261x^{46} + 74\,924\,784\,036\,765\,597\,482\,162\,224\,697\,378x^{45} +$$
$$33\,716\,152\,882\,551\,682\,431\,054\,950\,635\,828x^{44} + 14\,688\,225\,057\,065\,816\,000\,841\,247\,153\,422x^{43} +$$
$$6\,192\,094\,899\,403\,308\,142\,319\,324\,646\,830x^{42} + 2\,524\,931\,913\,311\,378\,421\,460\,541\,875\,013x^{41} +$$
$$995\,405\,857\,334\,028\,240\,446\,249\,995\,969x^{40} + 379\,202\,235\,047\,824\,659\,955\,968\,634\,895x^{39} +$$
$$139\,517\,805\,378\,058\,810\,895\,892\,716\,876x^{38} + 49\,548\,380\,452\,249\,950\,392\,015\,617\,673x^{37} +$$
$$16\,974\,908\,597\,922\,176\,404\,758\,662\,419x^{36} + 5\,606\,392\,061\,138\,587\,678\,507\,139\,578x^{35} +$$
$$1\,783\,852\,127\,215\,514\,388\,216\,575\,524x^{34} + 546\,405\,205\,018\,625\,434\,948\,486\,100x^{33} +$$
$$160\,994\,412\,344\,908\,368\,725\,437\,163x^{32} + 45\,591\,347\,244\,850\,943\,472\,027\,532x^{31} +$$
$$12\,397\,650\,890\,304\,440\,505\,241\,198x^{30} + 3\,234\,171\,338\,137\,153\,259\,094\,292x^{29} +$$
$$808\,543\,433\,959\,017\,353\,438\,195x^{28} + 193\,497\,799\,414\,541\,699\,555\,587x^{27} + 44\,275\,002\,111\,552\,677\,715\,575x^{26} +$$
$$9\,673\,558\,570\,858\,327\,142\,094x^{25} + 2\,015\,334\,387\,723\,540\,077\,262x^{24} + 399\,738\,890\,367\,674\,230\,448x^{23} +$$
$$75\,361\,590\,622\,423\,713\,249x^{22} + 13\,479\,601\,808\,118\,798\,229x^{21} + 2\,282\,919\,558\,918\,081\,919x^{20} +$$
$$365\,290\,003\,947\,963\,446x^{19} + 55\,089\,365\,597\,956\,206x^{18} + 7\,809\,413\,514\,931\,644x^{17} + 1\,037\,532\,923\,086\,353x^{16} +$$
$$128\,762\,751\,824\,308x^{15} + 14\,873\,235\,105\,552x^{14} + 1\,592\,645\,620\,686x^{13} + 157\,430\,569\,051x^{12} + 14\,303\,835\,837x^{11} +$$
$$1\,190\,322\,956x^{10} + 90\,590\,713x^{9} + 6\,345\,735x^{8} + 417\,209x^{7} + 27\,806x^{6} + 2\,024x^{5} + 216x^{4} + 26x^{3} + 5x^{2} + x + 1.$$

D

Rings and Algebras

D.1 Basic Definitions and Constructions

Definition 142 *A* (unitary) ring *is a triple* (R, α, μ) *where* (R, α) *is an abelian group whose operation* α *is written additively* $(\alpha(r, s) = r + s)$ *with neutral element* 0_R*, and* (R, μ) *is monoid, written multiplicatively* $(\mu(r, s) = r \cdot s)$ *with multiplicative neutral element* 1_R *such that these operations are coupled by* distributivity*, i.e.,* $(r + s) \cdot t = r \cdot t + s \cdot t, t \cdot (r + s) = t \cdot r + t \cdot s$ *for all* $r, s, t \in R$*. A ring is* commutative *iff its multiplicative monoid is commutative.*

A set map $f : R \to S$ *of rings* R, S *is a* ring homomorphism *iff it is a homomorphism of the underlying additive groups and a homomorphism of the underlying multiplicative monoids. The set of ring homomorphisms from* R *to* S *is again denoted by* $Hom(R, S)$ *if no confusion is likely.*

An element x *of a non-zero ring* R *is called* invertible*, iff there is a multiplicative* (*left*) *inverse* y*, i.e.,* $y \cdot x = 1_R$*. The subset* $R^\times$ *of invertible elements is a multiplicative group. A* skew field *is a ring such that* $R^\times = R - \{0_R\}$*. A* (commutative) field *is a commutative ring which is a skew field.*

The subring $Z(R)$ *of all elements in a ring* R *which commute with all of* R *is called the center of* R*. For a commutative* R*, a ring homomorphism* $\varphi : R \to S$ *whose image is in* $Z(S)$ *is called an* R-algebra*. If* φ *is clear, one says that "*S *is an* R*-algebra", and* φ *is called the "structural homomorphism". If* $\varphi : R \to S, \psi : R \to T$ *are two* R*-algebras, a ring homomorphism* $f : S \to T$ *is an* R*-algebra homomorphism, iff* $\psi = f \circ \varphi$*, i.e., iff the '*R*-elements are conserved under* f*'. One als often writes* r *instead of* $\varphi(r)$ *if the algebra structure is clear.*

The set-theoretic composition $g \circ f$ of two ring homomorphisms $f : R \to S, g : S \to T$ is again a ring homomorphism. A bijective ring homomorphism automatically has an *inverse* ring homomorphism, i.e., this is a ring *isomorphism*, and a ring *endomorphism* is a homomorphism with domain equal to its codomain, whereas an *automorphism* is an endomorphism which is an isomorphism. The corresponding concepts for R-algebras are evident: The homomorphisms have to be algebra homomorphisms, so, for example, an R-algebra automorphism is an automorphism which conserves the structural homomorphism.

Example 104 Classical examples of rings are the rings $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{H}$ of integers, rational numbers, real numbers, complex numbers and Hamilton quaternions. Except the integers, these rings are also skew fields, and $\mathbb{H}$ is not commutative. The conjugation $z = a + i.b \mapsto \bar{z} = a - i.b$ is an automorphism of the $\mathbb{R}$-algebra $\mathbb{C}$.

To every ring R, one has the opposite ring R^{opp} which is the same additive group, but multiplication is defined by reversing to given multiplication, i.e., $r \cdot^{opp} s = s \cdot r$. An anti-homomorphism of rings is a homomorphism into the opposite codomain ring.

Every ring R is a $\mathbb{Z}$-algebra in a unique way by $\varphi(z) = z.1_R$; the latter means that $z.1_R = 1_R + 1_R + \ldots 1_R$ if $z > 0$, it is 0_R if $z = 0$, and it is $-(-z).1_R$ if $z < 0$.

By the natural inclusions, $\mathbb{R}$ is a $\mathbb{Q}$-algebra, $\mathbb{C}$ is an $\mathbb{R}$-algebra, and $\mathbb{H}$ is an $\mathbb{R}$-algebra, but not a $\mathbb{C}$-algebra.

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The 2×2 matrices over a commutative ring R with coefficient-wise addition and usual matrix multiplication form a non-commutative ring. They are an R-algebra by the diagonal map $r \mapsto \left(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix}\right)$.

Proposition 86 *If I is a subgroup of a ring R, the following statements are equivalent:*

(*i*) *For all $r \in R$, $r \cdot I \subseteq I$, i.e., I is a* left ideal in R, *and $I \cdot r \subseteq I$,i.e., I is a* right ideal in R.
(*ii*) *There is a ring homomorphism $f : R \to S$ with $Ker(f) = I$.*

If I has the equivalent properties of Proposition 86, we call it an *ideal* or a *two-sided ideal* in R. So ideals correspond to normal subgroups. The *quotient ring* R/I (and the associated projection $\pi : R \to R/I$) is just the quotient group (with respect to the additive structure), together with the well-defined multiplication $(r+I)(s+I) = rs+I$, and we have $Ker(\pi) = I$. If G is any subset of a ring R, the smallest ideal containing G is denoted by (G) and is called the *ideal generated by* G.

Proposition 87 *Let I be an ideal in R, $\pi : R \to R/I$ the canonical homomorphism, and S any ring. Then*

$$Hom(R/I, S) \xrightarrow{\sim} \{f \in Hom(R,S) | I \subseteq Ker(f)\} : t \mapsto t \circ \pi$$

is a canonical bijection. If $I = Ker(f)$, then the corresponding morphism $R/I \to I$ is an isomorphism onto $Im(f)$.

Example 105 By the Euclidean algorithm (D.3), every subgroup of $\mathbb{Z}$ is of the form $\langle n \rangle = n.\mathbb{Z}$ for a uniquely determined non-negative n. Such a subgroup is also an ideal. The quotient ring $\mathbb{Z}_n = \mathbb{Z}/n.\mathbb{Z}$ is the *ring of integers modulo* n. It is the so-called "prime field of characteristic p" iff the Euler function $\phi(n) = n-1$, i.e., $n = p$ is a prime (see also Proposition 84 in Appendix C.3.4.1). For any skew field F, we have the unique $\mathbb{Z}$-algebra structure $p : \mathbb{Z} \to F$ whose kernel must be either (0) or a prime ideal $p.\mathbb{Z}$. In the latter case, the image $Im(p)$ is the smallest subfield in F; in the former case, the smallest subfield of F is evidently isomorphic to $\mathbb{Q}$. The prime field of a skew field is called its *prime field*, whereas the generator (zero or a positive prime) of $Ker(p)$ is called the characteristic $char(F)$ of the skew field.

A skew field F has only two ideals: $(0), F$, and conversely, a ring which has only these ideals is called *simple*. The ring $\mathbb{M}_{n,n}(F)$ of $n \times n$-matrices (see E.1.1) over a skew field F is simple (but not a skew field for $1 < n$).

D.1.1 Universal Constructions

If M is a multiplicative monoid and R is a commutative ring, we have a ring $R\langle M \rangle$, the *monoid algebra* by the following construction. The underlying set is the set $R^{(M)}$ of functions $f : M \to R$ with $f(m) = 0$ except of a finite number of arguments. This is an additive group under the addition $(f+g)(m) = f(m)+g(m)$. The product is defined by $(f \cdot g)(m) = \sum_{n \cdot n' = m} f(n) \cdot g(n')$, which is reasonable since the summands which do not vanish are finite in number. The multiplicative neutral element $1_{R\langle M \rangle}$ is the function $1_{R\langle M \rangle}(e_M) = 1_R$ and zero else. The elements $f : M \to R$ of the monoid algebra are also written as a formal sum $\sum_{f(m) \neq 0} f(m)m$. The algebra structure is given by $r \mapsto re_M$. Here is the *universal property* of the monoid algebra:

Proposition 88 *Every monoid homomorphism $\varphi : M \to (S, \mu)$ into the multiplicative monoid of an R-algebra S, can be extended in a unique way to a homomorphism $\Phi : S\langle M \rangle \to S$ of R-algebras. If $\sum f_m.m \in S\langle M \rangle$, its image is $\Phi(\sum f_m.m) = \sum f_m \cdot \varphi(m)$.*

A special case of a monoid algebra is for the free monoid $FM\langle Alphabet \rangle$ over *Alphabet*, see Example 96 in Appendix C.2.3. One usually writes $S\langle Alphabet \rangle$ instead of $S\langle FM(Alphabet) \rangle$. For example, if $Alphabet = \{X_1, X_2, \ldots X_n, \ldots\}$, we have the *algebra of non-commutative polynomials in the indeterminates* $X_1, X_2, \ldots X_n, \ldots$. If instead we take the free commutative monoid $FCM(Alphabet)$, we get the monoid algebra $S[Alphabet] = S\langle FCM(Alphabet) \rangle$ of commutative polynomials in the indeterminates $\{X_1, X_2, \ldots X_n, \ldots\}$ of *Alphabet*.

Example 106 For the R-algebra of polynomials in one variable $R[X]$, the ideal (X^2) defines the *R-algebra of dual numbers* $R[\varepsilon] = R[X]/(X^2)$ with $\varepsilon = X + (X^2)$ the class of X. An element of $R[\varepsilon]$ is uniquely described as a linear polynomial $a + \varepsilon.b$ in ε, and the multiplication is $(a + \varepsilon.b)(c + \varepsilon.d) = ac + \varepsilon.(ad + bc)$. The group $R[\varepsilon]^\times$ consists of those elements $a + \varepsilon.b$ with $a \in R^\times$.

Definition 143 *Let $(R_i)_{i\in I}$ be a family of rings, then the* product $\prod_{i\in I} R_i$ *of this family is the following ring: As a set, it is the product of the underlying sets, addition and multiplication are defined coordinate-wise*[1]*, i.e.,*

$$(x_i) + (y_i) = (x_i + y_i) \text{ and}$$
$$(x_i) \cdot (y_i) = (x_i \cdot y_i),$$

and the unity is the family (1_i) of unities 1_i in the respective rings R_i. For each index $j \in I$, we have a canonical projection ring homomorphism

$$p_j : \prod_{i\in I} R_i \to R_j$$

defined by $p_j((x_i)) = x_j$.

The product and its canonical projections shares the universal property of a product in the category of rings (see Appendix G.2.1 for the definition of a product in a category):

Lemma 80 *Let S be a ring, and let $(R_i)_{i\in I}$ be a family of rings. For each family $(f_i : S \to R_i)_{i\in I}$ of ring homomorphisms, there is a unique ring homomorphism $f : S \to \prod_{i\in I} R_i$ such that $f_i = p_i \cdot f$ for all $i \in I$.*

Clearly, the map $f(s) = (f_i(x))$ solves the problem.

More generally, consider a diagram[2] $\mathbf{D}$ of rings with vertex set I. Then we have a subring $lim(\mathbf{D})$ of the product $\prod_{i\in I} R_i$ consisting of all families (x_i) such that

$$f(x_j) = x_k \text{ for any homomorphism } f : R_j \to R_k$$

corresponding to a $\mathbf{D}$-arrow from vertex j to vertex k.

Proposition 89 *With the above notation, the ring of $lim(\mathbf{D})$, together with the induced canonical projections $p_i : lim(\mathbf{D}) \to R_i$, is the* limit *of the diagram $\mathbf{D}$ of rings, i.e., for any family $(f_i : S \to R_i)_{i\in I}$ of ring homomorphisms such that $f \cdot f_j = f_k$ whenever $f : R_j \to R_k$ is a homomorphism corresponding to a $\mathbf{D}$-arrow from vertex j to vertex k, there is a unique ring homomorphism $g : S \to lim(\mathbf{D})$ such that $f_i = p_i \cdot g$, all $i \in I$.*

In particular, if we have a fiber product[3] diagram $A \underset{f}{\to} C \underset{g}{\leftarrow} B$ of rings, there is a limit, the fiber product of this diagram, usually denoted by $A \times_C B$ and inserted in the commutative "pullback" square

$$\begin{array}{ccc} A \times_C B & \xrightarrow{p_2} & B \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & C \end{array} \tag{D.1}$$

of ring homomorphisms.

For non-commutative rings, fiber sums do not exist in general. However, if the diagram's homomorphisms are algebras over a commutative ring, we have the following well-known result [145, ch.III, No.2]:

[1] Addition and multiplication are taken in the respective rings.
[2] See definition 165, Appendix G.1.2, for a formal definition of a diagram in a category.
[3] See Appendix G.2.1 for the concept of a fiber product and the dual one of a fiber sum.

Theorem 52 *Let $A \underset{f}{\leftarrow} C \underset{g}{\rightarrow} B$ be a fiber sum diagram of algebras over the commutative ring C. Then the tensor product $A \otimes_C B$ defines a fiber sum*

$$\begin{array}{ccc} C & \xrightarrow{f} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i_1} \\ B & \xrightarrow{i_2} & A \otimes_C B \end{array} \tag{D.2}$$

of C-algebras with $i_1(a) = a \otimes 1_B$ and $i_2(b) = 1_A \otimes b$.

D.1.1.1 Quiver Algebras

Definition 144 *Suppose that we are given a quiver $Q = (\text{head}, \text{tail} : A \rightrightarrows V)$ (see Appendix C.2.2, definition 137). The* path category *of Q is the set $P(Q)$ of paths of Q (the morphisms of the category, the lazy paths being the objects of the category), together with the composition $p.q$ of two paths p, q, defined if $\text{head}(p) = \text{tail}(q)$. In that case, $p.q$ is just the evident composed path of length $l(p.q) = l(p) + l(q)$.*

Definition 145 *Suppose that we are given a finite quiver $Q = (\text{head}, \text{tail} : A \rightrightarrows V)$ and a commutative ring R. The quiver algebra $R\langle Q\rangle$ of Q with coefficients in R is the free left R-module whose basis is the path set $P(Q)$, together with R-bilinear extension of the composition in the path category, i.e., the product of two paths is their composition, if possible, and zero else. This means:*

$$\Big(\sum_i r_i.p_i\Big)\Big(\sum_j s_j.q_j\Big) = \sum_{i,j}(r_i.s_j)p_i.q_j.$$

The unity

$$1_{R\langle Q\rangle} = \sum_{v\in V} v$$

is the sum of all vertexes.

Definition 146 *Let Q be a quiver, and $R\langle Q\rangle$ its quiver algebra over the coefficient ring R. A sub-path of a path p in $R\langle Q\rangle$, is a triple (u, w, v) of paths such that $p = u.w.v$. If the external factors u, v are clear, the sub-path is identified with the middle member w, and we write $w \sqsubset p$.*

For example, if $l(p) = 0$ (a vertex), the only sub-path of p is p, in fact, $p = p.p.p$ is the only factorization of this kind. If $p = \circlearrowright x$ is a loop, the sub-paths of p are $x.x.p$, $p.x.x$, and $x.p.x$. So the middle member x appears two times since it is in different positions in the factorization.

If $p = u_1.w_1.v_1 = u_2.w_2.v_2$ are two subpaths of p with $l(w_1) = l(w_2)$, we write $(u_1, w_1, v_1) < (u_2, w_2, v_2)$ iff $l(u_1) < l(u_2)$. Clearly this relation is transitive and antisymmetric. Among all subpaths (u, w, v) of p with fixed middle length $l(w) = const.$, the relation corresponding to relation $\leqslant$ is an linear ordering. In fact, if $(u_1, w_1, v_1), (u_2, w_2, v_2)$ are two subpaths of p such that $l(u_1) = l(u_2)$ and $l(w_1) = l(w_2)$, then they are identical, so the ordering relation is total. We shall use this total ordering to define linear endomorphisms, so-called *sub-path operators* of the quiver algebras as follows.

We shall define an endomorphism $\sqsubset_i^{\lambda^i}$ for each natural index $i = 0, 1, 2, \ldots$ and for each system λ^i of coefficients in the following sense. We set

$$\lambda^i = (\lambda^i_{lj})_{i\leqslant l, 1\leqslant j\leqslant l-i+1}$$

with $\lambda^i_{lj} \in R$. The endomorphisms run as follows. Let w be a path of length $l(w) = l$. Then

$$\sqsubset_i^{\lambda^i}(w) = \begin{cases} 0 & \text{if } l < i, \\ \sum_{v_j \sqsubset w, l(v_j)=i} \lambda^i_{lj} v_j & \text{else,} \end{cases}$$

with the indexes of the subpaths of w referring to their total order defined above. In the musical applications, we shall encounter linear combinations $\Phi = \sum_i \mu_i \sqsubset_i^{\lambda^i}$ of such weighted sub-path operators.

D.2 Prime Factorization

An integer $1 < p$ is called *prime* iff $p = u \cdot v$ with positive factors implies $u = 1$ or $v = 1$.

Theorem 53 *Every non-zero integer x is a product*

$$x = \pm 1 \cdot p_1^{n_1} \cdot p_2^{n_2} \cdot \ldots p_k^{n_k}$$

with an increasing sequence of primes $p_1 < p_2 < \ldots p_k$ and positive exponents n_i which are all uniquely determined.

Corollary 32 *Let $p_1 < p_2 < \ldots p_k$ be an increasing sequence of primes. For two sequences of rational numbers $q_1 < q_2 < \ldots q_k$, $r_1 < r_2 < \ldots r_k$, the equation*

$$p_1^{q_1} \cdot p_2^{q_2} \cdot \ldots p_k^{q_k} = p_1^{r_1} \cdot p_2^{r_2} \cdot \ldots p_k^{r_k}$$

implies the equality of the sequences of rational numbers.

This implies that the logarithms $ln(p)$ of primes are linearly independent (see E.2.1) in the rational vector space $\mathbb{R}_{[\mathbb{Q}]}$ (see E.1.1), a central fact for the construction of the Euler module of pitch systems (see Section A.2.3).

D.3 Euclidean Algorithm

Proposition 90 *Euclidean algorithm: Given a non-zero integer d, every integer x has a unique representation $x = a.d + b, 0 \leqslant b < d$.*

Lemma 81 *Let $n > 1$ be an integer. Then every positive integer x has a representation*

$$x = \sum_{i=0}^{t} x_i.n^i$$

with $0 \leqslant x_i < n$, and $x_t \neq 0$. The x_i are uniquely determined. The representation is usually written as

$$x = x_t x_{t-1} \ldots x_1 x_0$$

and known as the n-adic representation.

D.4 Approximation of Real Numbers by Fractions

Lemma 82 *Let $L = \log_2(3)$. Then for every real number $\delta > 0$, there is a pair n, m of integers such that*

$$0 < n + m \cdot L < \delta.$$

Proof. We construct a sequence $(n_1, m_1), (n_2, m_2), \ldots (n_s, m_s), \ldots$ of pairs such that $0 < n_s + m_s \cdot L < 1/2^s$. We may start by $(n_1, m_1) = (2, -1)$ since $L \approx 1.58$. Suppose that we have found (n_s, m_s) such that $0 < n_s + m_s \cdot L < 1/2^s$. There is a maximal positive integer k such that $k \cdot (n_s + m_s \cdot L) < 1$. Then $(k + 1) \cdot (n_s + m_s \cdot L) > 1$ since L is not rational by Corollary 32. For the same reason, either

$$(k + 1) \cdot (n_s + m_s \cdot L) - 1 < 1/2 \cdot 1/2^s = 1/2^{s+1}$$

or

$$1 - k \cdot (n_s + m_s \cdot L) < 1/2^{s+1}.$$

Then either

$$(n_{s+1}, m_{s+1}) = ((k + 1) \cdot n_s - 1, (k + 1) \cdot m_s)$$

or

$$(n_{s+1}, m_{s+1}) = (1 - k \cdot n_s, -k \cdot m_s)$$

solves the problem, QED.

D.5 Some Special Issues

D.5.1 Integers, Rationals, and Real Numbers

Definition 147 *The index function* $index : \mathbb{R} \to \mathbb{Z}$ *is defined by*

$$index(x) = \begin{cases} 1 & \text{if } 0 < x, \\ -1 & \text{if } 0 > x, \\ 0 & \text{if } 0 = x \end{cases} \tag{D.3}$$

for $x \in \mathbb{R}$.

A real number x has unique representation $x = bottom(x) + x_+$ with $bottom(x) \in \mathbb{Z}, 0 \leqslant x_+ < 1$; we set $top(x) = floor(x) + 1$.

Definition 148 *The rounding function* $round : \mathbb{R} \to \mathbb{Z}$ *is defined by*

$$round(x) = \begin{cases} floor(x) & \text{if } x_+ \leqslant 0.5, \\ top(x) & \text{else} \end{cases} \tag{D.4}$$

for $x \in \mathbb{R}$.

E

Modules, Linear, and Affine Transformations

E.1 Modules and Linear Transformations

Definition 149 *Let R be a ring, then a* (left)[1] R-module *is a triple $(R, M, \mu : R \times M \to M)$ where M is an additively written abelian group and μ is the* scalar multiplication*, usually written as $\mu(r, m) = r.m$ if μ is clear (R is also called the ring of* scalars*, and M the group of* vectors*), with the properties:*

1. *We have $1_r.m = m$ for all $m \in M$.*
2. *For all $r, s \in R$ and $m, n \in M$, we have $(r+s).m = r.m+s.m$, $r.(m+n) = r.m+r.n$, and $r.(s.m) = (r \cdot s).m$.*

If $(R, M, \mu : R \times M \to M), (R, N, \nu : R \times N \to N)$ are two R-modules, a group homomorphism $f : M \to N$ is called R-linear (or a module homomorphism if the rest is clear) *iff it is "homogeneous", i.e., $f(r.m) = r.f(m)$, for all $r \in R, m \in M$, with the respective scalar multiplications. The set of R-linear homomorphisms from M to N is denoted by $Lin_R(M, N)$. It is an additive group under the pointwise addition $(f + g)(m) = f(m) + g(m)$. The set-theoretic composition $g \circ f$ of two module homomorphisms $f : M \to N, g : N \to L$ is also a module homomorphism, and we have distributivity, i.e., $(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$ for $g_i : N \to L, f : M \to N$ and $g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$ for $f_i : M \to N, g : N \to L$.*

By the distributivity of composition of module homomorphisms, the group $End_R(M) = Lin_R(M, M)$ is a ring, the *endomorphism ring of* M, which contains the multiplicative *automorphism group* $Aut_R(M) = End_R(M)^\times$ *of* M.

An R-linear homomorphism $f : M \to N$ has a group-theoretic kernel $Ker(f)$ and an image $Im(f)$ which are also submodules. For a submodule $N \subseteq M$, the quotient group M/N is also an R-module by the scalar multiplication $r.(m + N) = r.m + N$. The group-theoretic results Proposition 82 and the isomorphism Theorems 47, 48 in Appendix C, are valid literally if we replace the respective groups by modules (the normality of subgroups is automatic here).

E.1.1 Examples

Abelian groups G are canonically identified to $\mathbb{Z}$-modules by $z.g = g + g + \ldots g$, z times for $z > 0$, $(-z).g = -(z.g)$, and $0.g = 0_G$.

If M is an R-module, a submodule $N \subseteq M$ is a subgroup that is stable under scalar multiplication, i.e., $R.N = N$. If $S \subset M$ is a subset, the smallest submodule containing S is denoted by $\langle S \rangle$ and consists of all *linear combinations* $\sum_i r_i.s_i, r_i \in R, s_i \in S$. If there is a finite set S such that $\langle S \rangle = M$, M is called *finitely generated*. Finitely generated $\mathbb{Z}$-modules are completely classified, see C.3.4.2. If $(M_i)_{i \in I}$ is a family of submodules of M, we denote by $\Sigma_I M_i$ the module $\langle \bigcup_I M_i \rangle$. It consists of all finite sums $x_{i_1} + \ldots x_{i_k}, x_{i_j} \in M_{i_j}$.

Every ring R is a left R-module ${}_RR$ and a right R-module R_R by the given multiplication. For commutative rings these structures coincide. A left, right ideal in R identifies to a submodule of ${}_RR$, R_R, respectively.

[1] Right modules are defined in complete analogy, the scalar multiplication being written $m.r$ instead of $r.m$.

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If $\varphi : S \to R$ is a ring homomorphism and M is an R-module, we have an S-module structure $M_{[\varphi]}$ on M via $s.m = \varphi(s).m$, the *module defined by restriction of scalars*. If φ is clear, one also writes $M_{[S]}$. For the $\mathbb{Z}$-algebra structure of every ring R, $M_{[\mathbb{Z}]}$ gives the underlying structure of an abelian group M back. In particular, an R-algebra S is an R module via $({}_SS)_{[R]}$.

A *dilinear homomorphism* from an S-module M to an R-module N is a pair $(\varphi : S \to R, f : M \to N_{[\varphi]})$ consisting of a scalar restriction φ and an S-linear homomorphism f. If $(\varphi : S \to R, f : M \to N_{[\varphi]}), (\psi : T \to S, g : L \to M_{[\psi]})$ are two dilinear homomorphisms, their composition is defined by $(\varphi \circ \psi : T \to R, f \circ g : L \to N[\varphi \circ \psi])$. The set of dilinear homomorphisms from M to N is denoted by $Dil(M, N)$. If the scalar restriction is fixed by φ, we denote the corresponding set by $Dil_\varphi(M, N)$, and the special case $Dil_{Id_R}(M, N)$ is just $Lin_R(M, N)$ as above.

For any family $(M_i)_{i\in I}$ of R-modules, we have the *product module* $\prod_I M_i$. This is the product of the underlying groups, together with coordinatewise scalar multiplication $r.(m_i) = (r.m_i)$ The submodule of those (m_i) with only finitely many $m_i \neq 0$ is the *direct sum module* $\bigoplus_I M_i$. For every index j, one has the canonical (linear) projections $\pi_j : \prod_I M_i \to M_j$ and $\pi_j : \bigoplus_I M_i \to M_j$, via $\pi_j((m)_i) \mapsto m_j$, as well as the canonical (linear) injections $\iota_j : M_j \to \bigoplus_I M_i$ with $\iota_j(m)$ having zero coordinates except for coordinate index j where the value is m.

Lemma 83 (Universal limit property of direct products of modules) *For every family $(f_i : X \to M_i)_i$ of linear homomorphisms there is exactly one linear homomorphism $f : X \to \prod_I M_i$ such that $f_j = \pi_j \circ f$ for all $j \in I$.*

(Universal colimit property of direct sums of modules) *For every family $(f_i : M_i \to X)$ of linear homomorphisms there is exactly one linear homomorphism $f : \bigoplus_I M_i \to X$ such that $f_j = \circ f \circ \iota_j$ for all $j \in I$.*

A sum $\Sigma_I M_i$ of submodules M_i of a module M is called (inner) direct, iff the linear homomorphism $\bigoplus_I M_i \to M$ which is induced by the inclusions $M_i \subseteq M$ is an isomorphism.

For two positive integers m, n denote by $m \times n = [1, m] \times [1, n]$ the set of all pairs $(i, j), 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$. For a ring R, we have the direct sum $\mathbb{M}_{m,n}(R) = \bigoplus_{m\times n} R$ whose elements are written in the *matrix* notation

$$(r_{i,j}) = \begin{pmatrix} r_{1,1} & \dots & r_{1,n} \\ \dots & r_{i,j} & \dots \\ r_{m,1} & \dots & r_{m,n} \end{pmatrix}$$

whose *rows* or *columns* are the submatrices with constant first or second index, respectively.

If $(r_{i,j}) \in \mathbb{M}_{m,n}(R)$ and $(s_{j,k}) \in \mathbb{M}_{n,l}(R)$, we have the *matrix product* $(r_{i,j}) \cdot (s_{j,k}) = (t_{i,k}) \in \mathbb{M}_{m,l}(R)$ with $t_{i,k} = \sum_j r_{i,j} \cdot s_{j,k}$. Whenever defined, the product is associative. It is also distributive, i.e., $((r_{i,j}) + (r'_{i,j})) \cdot (s_{j,k}) = (r_{i,j}) \cdot (s_{j,k}) + (r'_{i,j}) \cdot (s_{j,k})$, and $(r_{i,j}) \cdot ((s_{j,k}) + (s'_{j,k})) = (r_{i,j}) \cdot (s_{j,k}) + (r_{i,j}) \cdot (s'_{j,k})$. For $m = n$, one has the identity matrix $E_m = (\delta_{ij})$ with the *Kronecker delta* $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$. With this identity and the matrix addition and multiplication, $\mathbb{M}_{m,m}(R)$ is a ring. With the matrix multiplication as scalar multiplication, $\mathbb{M}_{m,n}(R)$ becomes a left $\mathbb{M}_{m,m}(R)$-module and a right $\mathbb{M}_{n,n}(R)$-module.

If R is commutative, $\mathbb{M}_{m,m}(R)$ is an R-algebra via $r \mapsto r.E_m$, i.e., the scalar multiplication of the R-module $\mathbb{M}_{m,m}(R)$ coincides with the multiplication with R-elements from the algebra embedding.

E.2 Module Classification

E.2.1 Dimension

For any set C and ring R, we have the *free R-module* $\bigoplus_C R = R^C$ *of rank* $card(C)$ which is the direct sum of C copies of ${}_RR$ (for $C = \varnothing$, we take the zero module 0_R). A *free module* M is one that is isomorphic to a free module R^C. It is well known that the rank $card(C)$ is then uniquely determined and called the *dimension* $dim(M)$ of the free module M. If $R = F$ is a skew field, an F-module is called a *vector space* (over F), and we have the main fact of linear algebra:

Theorem 54 *Every vector space M over the skew field F is free, and the dimensions are a complete system of invariants of isomorphism classes of vector spaces.*

The proof of this theorem is based on the concept of *linear (in)dependence* in a module. A family $(m_i)_i$ of elements $m_i \in M$ is called *linearly independent* iff any (finite) linear combination $0 = \sum_{j=1,\dots k} r_j.m_{i_j}$ implies $r_j = 0$, for all j. Otherwise the family is called *linearly dependent*. A *base* of a module is a family of linearly independent elements which generates the module. The main Theorem 54 is proved by the *exchange theorem* which states that any family $(m_i)_i$ of linearly independent vectors can be inserted in a given basis by exchanging some of its elements with the $(m_i)_i$.

Example 107 $dim(\mathbb{R}_{[\mathbb{Q}]}) = card(\mathbb{R}) = 2^{\aleph_0}$, and the sequence of b-logarithms $(\log_b(p))_{p=\text{ prime}}$ is linearly independent by Corollary 32 in Appendix D.2. This means that for any finite increasing sequence $p. = (p_1, p_2, \dots p_k)$ of primes and the corresponding sequence

$$H_{p.} = (\log_b(p_1), \log_b(p_2), \dots \log_b(p_k)),$$

the scalar product map

$$H : \mathbb{Q}^k \to \mathbb{R}_{[\mathbb{Q}]}$$

with $H(x) = (H_{p.}, x) = \sum_i \log_b(p_i)x_i$ is a linear injection. The special case of the first three primes and $H_{prime} = (\log_b(2), \log_b(3), \log_b(5))$ was discussed in section A.2.3. Here we can prove the result needed for Proposition 76:

Lemma 84 *For any positive real bound δ and every real number ϕ, there is $x \in \mathbb{Z}^3$ such that $|H(x) - \phi| < \delta$*

Proof. WLOG, we can work with logarithm basis $b = 2$. We know from Lemma 82 in Appendix D.4 that there is $x' = (n, m, 0) \in \mathbb{Z}^3$ such that $0 < H(x') < \delta$. Clearly, there is an integer multiple $x = z.x'$ which does the job, QED.

Corollary 33 (of Theorem 54) *If $f : N \tilde{\to} R^n$ and $g : M \tilde{\to} R^m$ are two free modules of finite ranks n, m, isomorphic to the free modules R^n, R^m via isomorphisms f, g, then the linear homomorphisms are described by matrices: If $h : N \to M$ is a linear homomorphism, there is a uniquely determined matrix $H \in \mathbb{M}_{m,n}(R)$ such that for $x \in N$, we have $h(x) = g^{-1}(H \cdot f(x))$, where $f(x)$ is written as a column matrix in $\mathbb{M}_{n,1}(R)$ which canonically identifies to R^n. And conversely, each such matrix defines a linear homomorphism. In other words, there is an isomorphism*

$$Lin_R(N, M) \tilde{\to} \mathbb{M}_{m,n}(R) \tag{E.1}$$

of additive groups.

If R is a commutative ring, then an R-algebra S is also an R-module. We have an injective homomorphism of R-algebras

$$\begin{aligned} \Lambda : S &\to End_R(S) \\ \Lambda(s)(s') &= s \cdot s' \end{aligned} \tag{E.2}$$

which is called the *left regular representation of S*. If S is a free R-module of dimension n, then the isomorphism (E.1) induces the left regular representation in matrices:

$$\lambda : S \to \mathbb{M}_{n,n}(R). \tag{E.3}$$

More generally, a *linear representation* of an R-algebra A is an algebra homomorphism $f : A \to End_R(M)$ into the R-algebra of endomorphisms of an R-module M.

Definition 150 *The points of a k-element local composition K in $\mathbb{R}^n$ is* in general position *iff $dim(\mathbb{R}.K) = k - 1$.*

Theorem 55 *Let $v_1, v_2, \dots v_n$ be n vectors in a $\mathbb{Q}$-module, and take two submodules G, H of dimensions $g = dim(G), h = dim(H)$. Suppose that the module which is generated by the vectors v_i, G, and H, has dimension $n+g+h$. Then we can have at most $n+g+h$ points in general position in the union $\bigcup_i v_i + G + H$.*

Proof. WLOG, one may suppose $v_1 = 0$ after a shift. Take bases $x_1, \dots x_g$, $y_1, \dots y_h$ of G, H, respectively. Then, the vectors $0, v_2, \dots v_n, v_1+x_1, v_1+x_2, \dots x_g, v_1+y_1, \dots v_1+y_h$ are in general position. Conversely, if the vectors $z_1, z_2, \dots z_m$ in the union $\bigcup_i v_i + G + H$ are in general position, then $dim(\mathbb{Q}.\{z_1, z_2, \dots z_m\}) = m-1$. But $\langle z_1, z_2, \dots z_m \rangle$ is contained in the module which is spanned by $v_2, \dots v_n$, G, and H, whose dimension is $n + g + h - 1$, i.e., $m - 1 \leqslant n + g + h - 1$, QED.

E.2.2 Endomorphisms on Dual Numbers

For a commutative ring R, we have the commutative R-algebra $R[\varepsilon]$ of dual numbers (see Example 106 in Appendix D.1.1). As an R-module, it has dimension 2 and is isomorphic to R^2 under the map $a+\varepsilon.b \mapsto (a, b)$. By the above Corollary 33, the R-linear endomorphism ring of $R[\varepsilon]$ identifies to the four-dimensional matrix ring $\mathbb{M}_{2,2}(R)$. In this situation, the left regular representation of $R[\varepsilon]$ is the homomorphism of R-algebras

$$\lambda : R[\varepsilon] \to \mathbb{M}_{2,2}(R)$$

with

$$\lambda(a + \varepsilon.b) = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix},$$

which represents the linear endomorphism of multiplication by $a + \varepsilon.b$. We have shown in section 29.6 that $\mathbb{M}_{2,2}(R)$ is generated by the four R-linear basis elements $\lambda(1_R), \lambda(\varepsilon), \alpha_+, \alpha_+ \cdot \lambda(\varepsilon)$, where α_+ is the sweeping orientation

$$\alpha_+ = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

and that the R-algebra $\mathbb{M}_{2,2}(R)$ identifies to the quotient

$$R\langle \lambda(\varepsilon), \alpha_+ \rangle / (\lambda(\varepsilon)^2, \alpha_+^2 - \alpha_+, \alpha_+ \cdot \lambda(\varepsilon) + \lambda(\varepsilon) \cdot \alpha_+ - \lambda(1 + \varepsilon))$$

of the polynomial R-algebra in the two non-commuting variables $\lambda(\varepsilon), \alpha_+$.

E.2.3 Semi-simple Modules

A module $M \neq 0_R$ which has no submodules except $0_R, M$ is called *simple*. A module M is called *semi-simple* iff it has the following equivalent properties:

Lemma 85 *Let M be an R-module. The following statements are equivalent:*

(*i*) *Every submodule of M is a sum of simple submodules.*
(*ii*) *M is the sum of simple submodules.*
(*iii*) *M is the direct sum of simple submodules.*
(*iv*) *Every submodule N of M is a direct summand (i.e., there is a submodule N' of M such that $M = N \oplus N'$.*

The following is immediate:

Lemma 86 *A linear homomorphism between simple R-modules is either an isomorphism or zero. Hence the endomorphism ring $End_R(M)$ of a simple module M is a skew field.*

For example, the left $\mathbb{M}_{m,m}(F)$-module $\mathbb{M}_{m,n}(F)$ and the right $\mathbb{M}_{n,n}(F)$-module $\mathbb{M}_{m,n}(F)$ of $m \times n$-matrices over a skew field F is semi-simple. For the left module, the n columns are the simple submodules, whereas for the right module, the m rows are the simple submodules. Moreover, the only left and right submodules of $\mathbb{M}_{m,n}(F)$ are the zero module and $\mathbb{M}_{m,n}(F)$. In particular, the only two-sided ideals in the ring $\mathbb{M}_{m,m}(F)$ are 0 and $\mathbb{M}_{m,m}(F)$, i.e., this ring is simple.

Theorem 56 (Wedderburn) *The semi-simple rings are the finite products $\prod_i \mathbb{M}_{m_i,m_i}(F_i)$ of matrix rings $\mathbb{M}_{m_i,m_i}(F_i)$ over skew fields F_i.*

If G is a finite group, and if K is a commutative field, we have defined the monoid algebra $K\langle G\rangle$ in D.1.1, it is called the group algebra in this case. Here are the semi-simple group algebras over commutative fields:

Theorem 57 (Maschke) *The group algebra $K\langle G\rangle$ is semi-simple iff $char(K) \nmid card(G)$.*

E.2.4 Jacobson Radical and Socle

Definition 151 *The intersection of all maximal submodules of an R-module $M \neq 0$ is called the* Jacobson radical of M, *it is denoted by $Rad(M)$.*

Sorite 13 *let M, N be R-modules. Then:*

(*i*) *For $f \in Lin_R(M,N)$, we have $f(Rad(M)) \subseteq Rad(N)$.*
(*ii*) *We have $Rad(M \oplus N) = Rad(M) \oplus Rad(N)$.*
(*iii*) *We have $Rad(M/Rad(M)) = 0$.*
(*iv*) *We have $Rad({}_RR).M \subseteq Rad(M)$.*
(*v*) *If M is semi-simple, then $Rad(M) = 0$.*
(*vi*) *If a submodule N of M has M/N semi-simple, then $Rad(M) \subseteq N$.*

For a ring R, one may look at its left radical $Rad({}_RR)$, or at its right radical $Rad(R_R)$. Fortunately, there is no difference in that:

Proposition 91 *The left and right radicals of a ring R coincide, $Rad({}_RR) = Rad(R_R)$, and this (two-sided) ideal $Rad(R)$ is the maximal ideal I which annihilates every semi-simple module M, i.e., $I.M = 0$.*

For a ring R, we set

$$J_r = \{r \in R | 1_R \notin r \cdot R\}, J_l = \{r \in R | 1_R \notin R \cdot r\}.$$

Proposition 92 *For a ring R the following conditions are equivalent:*

(*i*) *The quotient ring $R/Rad(R)$ is a skew field.*
(*ii*) *We have $J_r = Rad(R)$.*
(*iii*) *We have $J_l = Rad(R)$.*
(*iv*) *The set J_r is additively closed.*
(*v*) *The set J_l is additively closed.*

Definition 152 *A ring with the equivalent properties of Proposition 92 is called* local. *In particular, a commutative ring R is local iff it has a unique maximal ideal.*

The *length $l(M)$ of a module M* is the maximum length l of finite chains $0 \subsetneq N_1 \subsetneq N_2 \subsetneq \ldots N_l = M$ of submodules (if that maximum is ∞, we set $l(M) = \infty$). A module is called *indecomposable* iff it is not the direct sum of two proper submodules.

Lemma 87 *If M is an R-module with local endomorphism ring, then M is indecomposable. Conversely, let M be an indecomposable module of finite length l. Then $End_R(M)$ is a local ring, and the radical $S = Rad(End_R(M))$ is nilpotent, namely $S^l = 0$.*

Proposition 93 *Let $X \subset M, Y \subset N$ be two non-zero submodules of indecomposable R-modules M, N of finite lengths. If $F : M \to N, g : N \to M$ are linear maps which induce mutually inverse isomorphisms $f|X : X \xrightarrow{\sim} Y, g|Y = (f|X)^{-1} : Y \xrightarrow{\sim} X$, then f, g are isomorphisms.*

Proof. Since $f \cdot f$ restricts to the identity of X, so does any of its positive powers. So non of them can be the zero endomorphisms of M. But from Lemma 87 we know that $End(M)$ is local and that a non-nilpotent endomorphism must be invertible. A symmetric argument yields inversibility of $f \cdot g$ and therefore of both, f and g, QED.

Lemma 88 (Fitting's lemma) *Let $f : M \to M$ be a linear endomorphism of a module M of finite length. Then there is a positive power f^n and a direct decomposition $M = N \oplus Ker(F^n)$ such that f^n is an automorphism on N.*

Definition 153 *The socle $Soc(M)$ of a module M is the sum of all its simple submodules.*

Theorem 58 *For a module M of finite length, the following three statements are equivalent:*

(*i*) *M is semi-simple.*
(*ii*) *$Rad(M) = 0$.*
(*iii*) *$Soc(M) = M$.*

E.2.5 Theorem of Krull-Remak-Schmidt

Theorem 59 (Krull-Remak-Schmidt) *Let $M_1, \dots M_k, M'_1, \dots M'_l$ be modules with local endomorphism rings (in particular these modules are all non-zero). Suppose that the direct sums $\bigoplus_{i=1,\dots k} M_i$ and $\bigoplus_{j=1,\dots l} M_j$ are isomorphism of R-modules. Then $k = l$, and there is a permutation σ of the indices such that we have isomorphisms $M_i \xrightarrow{\sim} M'_{\sigma(i)}$ for all $i = 1 \dots k$.*

Corollary 34 *A module of finite length is a direct sum of indecomposable submodules in a unique way up to permutation and isomorphisms of the summands.*

This follows from Theorem 59 in view of Lemma 87, QED.

E.3 Categories of Modules and Affine Transformations

See Appendix G for a reference to category theory.

For an additive group M and an element $m \in M$, the *translation by m* is the set map $e^m : M \to M : x \mapsto e^m(x) = m + x$. The exponential notation is chosen because $e^? : M \to \mathfrak{S}_M$ is an injective group homomorphism. We denote by e^M the group of translations on M, a group which is isomorphic to M.

Definition 154 *For two rings R, S, an R-module M and an S-module N, a* diaffine homomorphism *f is a map of form $e^n \cdot f_0$, where e^n is a translation on N and $f_0 \in Dil(M, N)$. The set of diaffine homomorphisms $f : M \to N$ is denoted by $M@N$. If we fix the underlying scalar restriction $\varphi : R \to S$ and only take $f_0 \in Dil_\varphi(M, N)$, the corresponding set is denoted by $M@_\varphi N$. In particular, if $\varphi = Id_R$, we write $M@_R N$ for the set of (R-)affine homomorphisms.*

Sorite 14 *If R, S, T are rings and M, N, L are modules over these rings, respectively, we have the following facts:*

(*i*) *If* $f = e^n \cdot f_0 \in M@N$, *then* $n = f(0)$, *and* $f_0 = e^{-n} \cdot f$. *So the* translation part e^n *and the* dilinear part f_0 *are uniquely determined.*

(*ii*) *If* $f = e^n \cdot f_0 \in M@N, g = e^l \cdot g_0 \in N@L$, *then the set-theoretic composition* $g \cdot f \in M@L$ *and*

$$g \cdot f = e^{l+g(n)} \cdot g_0 \cdot f_0. \tag{E.4}$$

(*iii*) *The diaffine* $f = e^n \cdot f_0 \in M@N$ *is an isomorphism iff its dilinear part* f_0 *is so, and then the inverse is* $f^{-1} = e^{-f_0^{-1}(n)} \cdot f_0^{-1}$.

(*iv*) *If* $f, g \in M@_\varphi N$, *then so is the pointwise difference* $f - g$, *hence* $M@_\varphi N$ *is an additive group. If* S *is commutative,* $M@_\varphi N$ *is an* S*-module and also an* R*-module under* φ.

(*v*) *For* $M = N, \varphi = Id_R$, *the* general affine group $\overrightarrow{GL}(M)$, *i.e., the group of affine automorphisms is isomorphic to the semidirect product* $M \rtimes_\phi Gl(M)$ *of* M *with the* general linear group $GL(M)$ *of linear automorphisms of* M *under the group action* $\phi = Id_{GL(M)}$.

The category of modules and diaffine homomorphisms is denoted by **Mod**, whereas the subcategory of (left) R-modules and R-affine homomorphisms is denoted by $\mathbf{Mod}_R$. If $f : R \to S$ is a ring homomorphism, we have the scalar extension functor

$$S \otimes_R ? : \mathbf{Mod}_R \to \mathbf{Mod}_S : M \mapsto S \otimes_R M \tag{E.5}$$

which acts by scalar extension of the linear parts of morphisms and by the canonical map $M \to S \otimes_R M : x \mapsto 1 \otimes x$ on the translation part. See [145, II.5.1] for details.

In **Mod**, we have to add an additional object $\varnothing_R$ for each ring R. This is the empty set plus the unique possible scalar multiplication. This is *not* a module in the usual sense since it is not even a group! But there are important category-theoretic reasons to introduce these objects. Observe that for an S-module N, $\varnothing_R@N$ is in bijection with the set of ring homomorphisms $Hom(R, S)$, whereas $N@\varnothing_R$ is empty if N is not empty.

By $\mathbf{Mod}^{@}$, we denote the category of set-valued presheaves on **Mod**, i.e., the contravariant set-valued functors $F : \mathbf{Mod} \to \mathbf{Ens}$. In particular, the Yoneda embedding $\mathbf{Mod} \to \mathbf{Mod}^{@}$ yields the representable presheaf $@M$ for a module M, with $@M(X) = X@M$ for $X \in {}_0\mathbf{Mod}$. This is one reason why we also write $X@F$ for the evaluation $F(X)$, even if F is not representable. By $M@$ we denote the covariant functor with $M@(X) = M@X$. In the context of presheaves, we often call a module X that is an argument of such presheaves an *address*; the reasons for this wording are made explicit in the musicological Chapter 6 on forms and denotators.

E.3.1 Direct Sums

Proposition 94 *Let* A *be an* R*-module, and* n *a natural number. Then there is a canonical isomorphism* $@A^{\oplus n} \xrightarrow{\sim} (@A)^n$, *i.e.,* $A^{\oplus n}$ *represents the* n*-fold product functor.*

Proof. Let X be any module. Then every affine homomorphism $f = e^t \cdot f_0 : X \to A^{\oplus n}$ projects to the n factors $f_i = p_i \cdot f$ via the respective projections $p_i : A^{\oplus n} \to A$. Also, the dilinear part f_0 projects to the n dilinear factors $f_{0,i} : X \to A$. Let t_i be the i^{th} component of t. Then we have $f_i = e^{t_i} \cdot f_{0,i}$. This yields the desired bijection $X@A^{\oplus n} \xrightarrow{\sim} (X@A)^n$, and this is functorial in X. QED.

E.3.2 Affine Forms and Tensors

In this section we suppose that all modules have a commutative coefficient ring R, i.e., we work in the category $\mathbf{Mod}_R$. Tensor products[2] are automatically taken over R. By $X^\star$ we denote the R-linear dual $Lin_R(X; R)$ of the R-module X.

[2] See [145] for tensor products.

For the R-module $A^\star$ of affine forms on an R-module (address) A and an R-module M, we have a canonical linear injection $M \rightarrowtail A^\star \otimes M$. In fact, there is an R-linear isomorphism $A^\star \xrightarrow{\sim} R \oplus A^\star : e^r \cdot x \mapsto r+x$, and we deduce an R-linear isomorphism $A^\star \otimes M \xrightarrow{\sim} M \oplus A^\star \otimes M$, whence the above injection; it maps $m \in M$ to $e^{1_R}.0 \otimes m$.

With the above notation, fix an R-module A (an address). We have the subfunctor

$$A@_R : \mathbf{Mod}_R \to \mathbf{Ens} : M \mapsto A@_R M \tag{E.6}$$

of $A@$, and induced on the subcategory $\mathbf{Mod}_R$ of $\mathbf{Mod}$. We further have the functor

$$A^\star \otimes : \mathbf{Mod}_R \to \mathbf{Ens} : M \mapsto A^\star \otimes M. \tag{E.7}$$

This functor acts as follows on affine morphisms $F = e^n \cdot F_0 : M \to N$. Identify n with the canonically associated element of $A^\star \otimes N$. Then, we define

$$A^\star \otimes F = e^n \cdot A^\star \otimes F_0, \tag{E.8}$$

in other words, $(A^\star \otimes F)_0 = A^\star \otimes F_0$. For the composition

$$e^{t+G_0(n)} \cdot G_0 F_0 : M \xrightarrow{F} N \xrightarrow{G} T$$

of affine morphisms $F = e^n \cdot F_0$ and $G = e^t \cdot G_0$ this implies

$$\begin{aligned} A^\star \otimes GF =& e^{t+G_0(n)} \cdot A^\star \otimes G_0 F_0 \\ =& e^t \cdot e^{G_0(n)} \cdot A^\star \otimes G_0 \cdot A^\star \otimes F_0 \\ =& e^t \cdot A^\star \otimes G_0 \cdot e^n \cdot A^\star \otimes F_0 \\ =& A^\star \otimes G \cdot A^\star \otimes F, \end{aligned}$$

whence the claimed functoriality.

Lemma 89 *With the above notation, there is a natural transformation*

$$\theta : A^\star \otimes \to A@_R. \tag{E.9}$$

If M is an R-module, it is defined by its action on pure tensors $x \otimes m \in A^\star \otimes M$ by

$$\theta(x \otimes m) : A \to M : a \mapsto x(a)m. \tag{E.10}$$

The natural transformation θ is an isomorphism if A is a finitely generated projective module.

Proof. The formula (E.10) is an extension of a classical formula in the linear case, see [145, II.74]. In fact, write $A^\star \otimes M \xrightarrow{\sim} M \oplus A^\star \otimes M$, and then $A@M \xrightarrow{\sim} M \oplus Lin_R(A, M)$. Then the classical formula $\theta_0 : A^\star \otimes M \to Lin_R(A, M) : x \otimes m \mapsto \theta_0(x \otimes m)$ with $\theta_0(x \otimes m)(a) = x(a)m$ extends to the linear map

$$\theta'(m + \lambda) = m + \theta_0(\lambda)$$

which for special pure tensor arguments $x \otimes m = e^r \cdot x_0 \otimes m$ yields

$$\begin{gathered} \theta'(rm + x_0 \otimes m) = rm + \theta_0(x_0 \otimes m), \text{ corresponding to} \\ e^{rm} \cdot \theta_0(x_0 \otimes m), \end{gathered}$$

and the latter evaluates to

$$(e^{rm} \cdot \theta_0(x_0 \otimes m))(a) = rm + x_0(a)m = x(a)m$$

which means $\theta' = \theta$. Let us then prove the naturality of θ. Let $F = e^n \cdot F_0 : M \to N$ be an affine morphism. We have to show that the diagram

$$\begin{array}{ccc} A^\star \otimes M & \xrightarrow{\theta} & A@M \\ {\scriptstyle A^\star \otimes F}\downarrow & & \downarrow{\scriptstyle A@F} \\ A^\star \otimes N & \xrightarrow{\theta} & A@N \end{array} \tag{E.11}$$

is commutative. It suffices to verify it for pure tensors $x \otimes m \in A^\star \otimes M$. Take $a \in A$. Then

$$\begin{aligned} A@F(\theta(x \otimes m))(a) =& F(\theta(x \otimes m)(a)) \\ =& F(x(a)m) \\ =& n + F_0(x(a)m) \\ =& n + x(a)F_0(m) \\ =& \theta(n) + \theta(x \otimes F_0(m))(a) \\ =& \theta(n + x \otimes F_0(m))(a) \\ =& \theta((A^\star \otimes F)(x \otimes m))(a) \end{aligned}$$

and we are done with diagram (E.11). If A is finitely generated projective, the classical linear map θ_0 is iso, and hence so is θ. QED.

Observe that the special case $A = 0$ of zero address is included in the lemma and that in this case, θ identifies to the identity transformation on the forgetful functor.

E.3.3 Biaffine Maps

In this section we again suppose that all modules have a commutative coefficient ring R, i.e., we work in the category $\mathbf{Mod}_R$. In classical module theory, the tensor product is a universal construction relating to bilinear maps. The extension to biaffine maps runs as follows.

Definition 155 *Let U, V, W be modules in $\mathbf{Mod}_R$. A map*

$$f : U \times V \to W$$

is called biaffine *if it is affine in each variable, i.e., if*

$$\begin{aligned} &f_u : V \to W : v \mapsto f_u(v) = f(u, v) \text{ and} \\ &f^v : U \to W : u \mapsto f^v(u) = f(u, v) \end{aligned}$$

are all affine, i.e., $f_u \in V\,@_R W$ and $f^v \in U\,@_R W$. The set of all biaffine maps f is denoted by $\mathcal{A}_2(U, V; W)$.

Lemma 90 *For R-modules U, V, W in* **Mod**, *there is a canonical bijection*

$$\mathcal{A}_2(U, V; W) \xrightarrow{\sim} U\,@_R(V\,@_R W). \tag{E.12}$$

If these sets are given their canonical structure of R-modules, bijection E.12 is an isomorphism of R-modules.

Proof. Let $f : U \times V \to W$ be a biaffine map. Then the associated map $f_u, u \in U$ by definition stays in $V@_R W$. So we have a map $f_? : U \to V@_R W$. Let us show that this map is affine. Set $\lambda(v) = f(0, v)$, this affine map is the constant part of our candidate $f_?$, i.e., we claim that $f_u - \lambda$ is linear in u. But $f_u(v) - \lambda(v) = f^v(u) - \lambda(v) = f^v(u) - f^v(0)$ is linear in u and we are done. Conversely, each $g \in U@_R(V@_R W)$ defines $\tilde{g}(u, v) = g(u)(v)$ with $\tilde{g} \in \mathcal{A}_2(U, V; W)$, and this clearly is an inverse to the map $f \mapsto f_?$. That we have a module isomorphism is clear. QED.

Proposition 95 *Let U, V, W be R-modules in* **Mod**, *and define the* affine tensor product $U \boxtimes V = U \otimes V \oplus U \oplus V$. *Then we have a canonical bijection*

$$\boxtimes : \mathcal{A}_2(U, V; W) \xrightarrow{\sim} (U \boxtimes V) @_R W, \tag{E.13}$$

i.e., $U \boxtimes V$ is a universal object in the affine category $\mathbf{Mod}_R$ *like the tensor product is for R-linear maps. If $f \in \mathcal{A}_2(U, V; W)$, then its image $\boxtimes f$ applies a typical element $u \otimes v + r + s$ to*

$$\boxtimes f(u \otimes v + r + s) = {}_\square f(u \otimes v) + f^0(r) + f_0(s) - f(0,0) \tag{E.14}$$

where ${}_\square f$ is the linear map associated with the bilinear map

$$f_\square(u, v) = f(u, v) - f_0(v) - f^0(u) + f(0,0). \tag{E.15}$$

The universal map $i : U \times V \to U \boxtimes V$ is defined by $i(u, v) = u \otimes v + u + v$.

Proof. The proposition follows directly from Lemma 90, the definition of the affine tensor product and the universal property of the linear tensor product. We then have $\boxtimes f(i(u, v)) = \boxtimes f(u \otimes v + u + v) = {}_\square f(u \otimes v) + f^0(u) + f_0(v) - f(0,0) = f_\square(u, v) + f^0(u) + f_0(v) - f(0,0) = f(u, v) - f_0(v) - f^0(u) + f(0,0) + f^0(u) + f_0(v) - f(0,0) = f(u, v)$. QED.

Definition 156 *For modules U, V, W, X in* $\mathbf{Mod}_R$ *and affine maps $f : U \to W$, $g : V \to X$, the affine tensor product map*

$$f \boxtimes g : U \boxtimes V \to W \boxtimes X \tag{E.16}$$

is defined as the canonical affine map $\boxtimes h$ according to Proposition 95 which is associated with the biaffine map

$$h : U \times V \to W \boxtimes X : (u, v) \mapsto i(f(u), g(v)).$$

Sorite 15 *For modules U, V, W, X in* $\mathbf{Mod}_R$ *and with the notation of section E.3.8, we have:*

(i) $U \boxtimes V \xrightarrow{\sim} V \boxtimes U$.
(ii) $U \boxtimes 0_R \xrightarrow{\sim} 0_R \boxtimes U \xrightarrow{\sim} U$.
(iii) $U \boxtimes (V \boxtimes W) \xrightarrow{\sim} (U \boxtimes V) \boxtimes W$, *i.e., we can identify these products and write* $U \boxtimes V \boxtimes W$.
(iv) $(U \amalg V) \boxtimes W \xrightarrow{\sim} U \boxtimes W \amalg V \boxtimes W$.

For a module M in $\mathbf{Mod}_R$, the functor $@_R M : \mathbf{Mod}_R \to \mathbf{Ens} : B \mapsto B@_R M$ contains redundant structure in B since there are elements in B which are annihilated by all linear maps into M. We want to reduce B to a module where this annihilator set is the zero submodule.

Definition 157 *With the above notation, set $An(B, M) = \bigcap_{k \in B@M)} Ker(k_0)$, denote $B_{/M} = B/An(B, M)$ and write $/M : B \to B_{/M}$ for the canonical projection. The module $B_{/M}$ is called the* M-reduction of B.

The following lemma is clear:

Lemma 91 *The assignment $?_{/M} : B \mapsto B_{/M}$ defines a functor on* $\mathbf{Mod}_R$. *The projection $/M : B \to B_{/M}$ and the uniquely defined commutative diagrams*

$$\begin{array}{ccc} C & \xrightarrow{/M} & C_{/M} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_{/M}} \\ B & \xrightarrow{/M} & B_{/M} \end{array} \tag{E.17}$$

which are associated with affine homomorphisms $f : C \to B$ define a natural transformation on $Id_{\mathbf{Mod}_R} \to ?_{/M}$.

Proposition 96 *Let* $@_R^{red}M = @_R M\cdot ?_{/M}$ *Then we have a natural isomorphism* $@_R^{red}M \tilde{\to} @_R M$.

Proof. In fact, the natural transformation $/M : B \to B_{/M}$ induces an isomorphism (of R-modules) $B@_R^{red}M \tilde{\to} B@_R M$. QED.

Corollary 35 *With the above notation, the functors* $B\tilde{@}M$ *and* $B\tilde{@}^{red}M = B_{/M}\tilde{@}M$ *are canonically isomorphic.*

Proof. If $A \in {}_0\mathbf{Mod}_R$, we have

$$\begin{aligned} A@B\tilde{@}^{red}M &= A@B_{/M}\tilde{@}M \\ &= A\boxtimes B_{/M}@_R M \\ &\tilde{\to} A@_R(B_{/M}@_R M) \\ &\tilde{\to} A@_R(B@_R M) \\ &\tilde{\to} A\boxtimes B@_R M \\ &= A@B\tilde{@}M. \end{aligned}$$

Let M, A be modules in $\mathbf{Mod}_R$. Then we have

Lemma 92 *There is an isomorphism of R-modules*

$$Lin_R(M, A@_R R) \tilde{\to} A@_R M^\star \tag{E.18}$$

which is functorial in both, A and M.

Proof. We have these functorial isomorphisms:

$$\begin{aligned} Lin_R(M, A@_R R) &\tilde{\to} Lin_R(M, A^\star \oplus R) \\ &\tilde{\to} Lin_R(M, A^\star) \oplus M^\star \\ &\tilde{\to} (M\otimes A)^\star \oplus M^\star \\ &\tilde{\to} (A\otimes M)^\star \oplus M^\star \\ &\tilde{\to} Lin_R(A, M^\star) \oplus M^\star \\ &\tilde{\to} A@_R M^\star. \end{aligned}$$

Proposition 97 *Let M be as above, and $A = R^n$, $0 \leqslant n$, then we have canonical R-linear maps*

$$u : A@_R M^\star \to (A@_R M)^\star, \tag{E.19}$$
$$d : A@_R M \to (A@_R M^\star)^\star, \tag{E.20}$$

which are isomorphisms for M finitely generated and projective.

Proof. As to the first map, a linear map $v : Lin_R(A, M^\star) \to Lin_R(A, M)^\star$ is defined as follows: For $g : A \to M^\star$ and $f : A \to M$, we have the composition $f^\star \cdot g : A \to A^\star$, which is a bilinear form on A, and we may set $e(g)(f) = tr(f^\star \cdot g)$, a linear function of g, calculated in the canonical bases of A and $A^\star$. This map is canonically extended by the identity on M and we are done for u. For the second map, we have the canonical bidual linear map $l : A@_R M \to A@_R M^{\star\star}$, and we may apply the first map to the bidual of M. The statement concerning finitely generated projective modules is standard. QED

E.3.4 Symmetries of the Affine Plane

We consider symmetries, i.e., affine transformations $D = e^t \cdot H$ on $\mathbb{R}^2$. From the geometric point of view, the set $\mathbb{R}^2@\mathbb{R}^2$ of these maps is described as follows. Fix a (zero-addressed) local composition $\Delta = \{u, v, w\}$ in the real plane $\mathbb{R}^2$, with three points in general position (see Appendix E.2.1). Then we know from section 15.2.1 that the map

$$B(\Delta) : \mathbb{R}^2@\mathbb{R}^2 \to (\mathbb{R}^2)^3 : D \mapsto (D(u), D(v), D(w)) \tag{E.21}$$

is a bijection. We use this bijection to describe some special transformations:

Shearings. Let G be a straight line in $\mathbb{R}^2$ (not necessarily through the origin), and let u, v, w be in general position such that u, v lie on G, whereas w does not. A *shearing S relating to G* is a symmetry which leaves both, u, v, fixed and transforms w into $w + r.(v - u), r \in \mathbb{R}$. Then G remains fixed identically, and w is shifted in parallel motion with respect to G. The n^{th} power S^n of S is the shearing which fixes G and transforms w into $w + nr.(v - u)$.

Dilatations. For a point $u \in \mathbb{R}^2$, and two scalars $\delta, \sigma \in \mathbb{R}$, a *dilatation D by factors δ, σ* and centered in u is defined by the prescription that $D(u) = u$, and that for two other points v, w such that u, v, w are in general position, we have $D(v) = u + \delta.(v - u), D(w) = u + \sigma.(w - u)$. A dilatation with $\delta = \sigma = -1$ is called *point reflection with center u*, it corresponds to a rotation by 180° around u.

Glide Reflections. Let G be a straight line, and take u, v, w as in the above paragraph E.3.4 about shearing. A *glide reflection P* is a symmetry, such that $P(u)$ lies on G, and $P(v) - P(u) = v - u, P(w) - P(u) = u - w$. Therefore, $P(v)$ lies also on G, and $P(w)$ lies on the "opposite side" of w on the line through w, u, i.e., P is a translation by $P(u) - u$, followed by a 'skew' reflection in G in the direction of the line through w and u. Especially for the diagonal $G = \mathbb{R}.(1, 1)$ and $u = P(u) = 0, w = (-1, 1)$, we obtain the exchange of coordinate axes: the *parameter exchange*.

E.3.5 Symmetries on $\mathbb{Z}^2$

Theorem 60 *Every symmetry $f \in \mathbb{Z}^2@\mathbb{Z}^2$ is the product of some of the following symmetries:*

1. *a translation $T = e^{(0,1)}$,*
2. *a shearing S which leaves the first axis $\mathbb{Z}.(1, 0)$ fixed and transforms $(0, 1)$ to $(1, 1)$,*
3. *the parameter exchange P,*
4. *the reflection K at the second axis,*
5. *the dilatations $D_m, 0 \leqslant m$ in the direction of the first axis by factor m.*

The general affine group $\overrightarrow{GL}(\mathbb{Z}^2)$ is generated by T, S, P, K.

Proof. The statement concerning the general affine group is immediate from the first part of the theorem. To show the latter, we say that a symmetry X is "good" if X can be written as a product of symmetries of the required shape. Here are the matrices for the generators:

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, D_m = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}.$$

We first show that all 2×2-matrices, i.e., all linear maps $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are products of symmetries of type S, K, D_m, P. Observe that $P^2 = K^2 = E_2$.

1. If $P \cdot X$ is good, then so is $X = P^2 \cdot X = P \cdot (P \cdot X)$. The same is valid for $X \cdot P$. Here, $P \cdot X$ is the exchange of rows in X, whereas $X \cdot P$ is the exchange of columns in X.

2. We have $S^{-1} = K \cdot S \cdot K = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Therefore the powers $S^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ are good for integer n. Further, $P \cdot S^{\pm 1} \cdot P = \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$ and therefore $(P \cdot S \cdot P)^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ is good.
3. For $m \geqslant 0$, $P \cdot D_m \cdot P = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$ and $P \cdot K \cdot D_m \cdot P = \begin{pmatrix} 1 & 0 \\ 0 & -m \end{pmatrix}$ are good.
4. If a coefficient of X vanishes, one can enforce $c = 0$ by a row or column exchange. Because of
$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$
and the preceding results, such an X is also good.
5. If no coefficient of X vanishes, one can apply $\pm E_2$ and exchange of columns that satisfy $c \geqslant d > 0$. Applying the Euclidean algorithm (section D.3), we can write $c = nd + r, 0 \leqslant r < d$. This yields
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix} = \begin{pmatrix} a - nb & b \\ c - nb & d \end{pmatrix} = \begin{pmatrix} a' & b \\ r & d \end{pmatrix}.$$
If $r = 0$, we are in case 4 above. Else, we have $d > r > 0$. Via column exchange we obtain $\begin{pmatrix} b & a' \\ d & r \end{pmatrix}$, and we may set $c' = d, d' = r$. But $d' < d$, so the algorithm leads to case 4 after a finite number of steps. This settles the linear maps.

For translations, observe the following identities:

$$e^{(1,0)} = P \cdot T \cdot P, e^{(-1,0)} = K \cdot e^{(1,0)} \cdot K, e^{(0,-1)} = P \cdot e^{(-1,0)} \cdot P.$$

Therefore all transpositions are good since we have natural numbers x, y such that $e^{(\pm x, \pm y)} = (e^{(\pm 1,0)})^x \cdot (e^{(0,\pm 1)})^y$. This settles all the cases, QED.

E.3.6 Symmetries on $\mathbb{Z}^n$

For integers $n \geqslant 2$ and $1 \leqslant i < j \leqslant n$, we have the *diagonal embedding* $\Delta_{i,j} : GL(\mathbb{Z}^2) \rightarrowtail GL(\mathbb{Z}^n)$ defined by

$$\Delta_{i,j}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (x_{u,v}) \text{ with}$$
$$x_{i,i} = a, x_{i,j} = b, x_{j,i} = c, x_{j,j} = d,$$
$$x_{u,u} = 1 \text{ for } u \neq i, j, \text{ and}$$
$$x_{u,v} = 0 \text{ else.}$$

Theorem 61 *For an integer $n \geqslant 2$ the group $GL(\mathbb{Z}^n)$ is generated by the diagonal embedded groups $\Delta_{1,j} GL(\mathbb{Z}^2), 1 \leqslant j \leqslant n$.*

The proof goes by induction on n and uses the Euclidean algorithm, we leave it as an exercise.

E.3.7 Complements on the Module of a Local Composition

Lemma 93 *Let A be an address module over the commutative coefficient ring R and $(K, A@M)$ a commutative local composition. Then:*

(i) $R.K \subset \langle K \rangle$.
(ii) $R.K = \langle K \rangle$ *iff* K *is embedded.*
(iii) *Let* $f : K \to L$ *is a morphism of embedded commutative local compositions*

$$(K, A@_R M), (L, A@_R N)$$

at the same address A. *Then any underlying symmetry* $F : M \to N$ *restricts to* $R.K$ *and* $R.L$, *i.e.,* $F(R.K) \subset R.L$, *and this restriction is uniquely determined by* f. *We denote this affine map by* $R@f : R.K \to R.L$.

Proof. The first statement is clear since $R.K = \langle x - x_0 |\ x \in K \rangle$. If $R.K = \langle K \rangle$, then obviously $K \subset \langle K \rangle = R.K$, and K is embedded. Conversely, if $K \subset R.K$, then also $\langle K \rangle \subset R.K$ and equality follows from (i). As to (iii), observe that we have a linear application $R.f : R.K \to R.L$ which is induced by the linear part F_0 of F, and which is only a function of f, by Lemma 6 of Chapter 8. Further, if $F = e^n \cdot F_0$, and if $k \in K$, we have $n = F(k) - F_0(k) = f(k) - R.f(k)$ since $K \subset R.K$; in other words, $n = n_f$ is only a function of f, and not of the underlying F. Therefore, since both, $f(k)$ and $R.f(k)$, are elements of $R.L$, $n_f \in R.L$. This means that for $x \in R.K$, $F(x) = n_f + F_o(x) = n_f + R.f(x) = R@f(x) \in R.L$, QED.

E.3.8 Fiber Products and Fiber Sums in Mod

Theorem 62 *The category* **Mod** *of modules and diaffine transformations has arbitrary fiber products.*

Proof. We are given a fiber product diagram

$$K \underset{f}{\to} M \underset{g}{\leftarrow} L \tag{E.22}$$

of modules over the fiber product diagram

$$A \underset{u}{\to} C \underset{v}{\leftarrow} B \tag{E.23}$$

of corresponding coefficient rings. If any of these modules K, L or M is empty, or if intersection $\mathrm{Im}(f) \cap \mathrm{Im}(g)$ is empty, then the empty module over the fiber product $A \times_C B$ of coefficient rings does the job. So we may suppose that neither of these four spaces is empty.

Consider the dilinear parts f_0 and g_0 of f and g. Then we have the dilinear homomorphism $d : K \oplus L \to M : (k, l) \mapsto f_0(k) - g_0(l)$ with regard to the fiber product ring homomorphism $A \times_C B \to C$. Take any couple $(k, l) \in K \oplus L$ with $f(k) = g(l)$. Then the set-theoretic fiber product $\Delta \subset K \oplus L$ equals $Ker(d) + (k, l)$. This implies that the diaffine embedding $e^{(k,l)} : Ker(d) \xrightarrow{\sim} \Delta \subset K \oplus L$, followed by the projections to K and L defines a fiber product

$$\begin{array}{ccc} Ker(d) & \xrightarrow{p_2 \cdot e^{(k,l)}} & L \\ {\scriptstyle p_1 \cdot e^{(k,l)}}\downarrow & & \downarrow{\scriptstyle g} \\ K & \xrightarrow{\quad f \quad} & M \end{array} \tag{E.24}$$

of modules and diaffine transformations. QED.

Theorem 63 *The category* **Mod** *has fiber sums for all pushout diagrams of modules over a fixed coefficient ring, i.e., where the scalar restrictions are the identity.*

Proof. We are given a fiber sum diagram

$$K \underset{f}{\leftarrow} M \underset{g}{\to} L \tag{E.25}$$

of modules over coefficient ring A. If M is empty we have to construct the sum $K \amalg L$, and we may suppose that both summands are nonvoid, the other cases being trivial. Consider the direct sum $S = K \oplus L \oplus A$ and the affine injections

$$i_1 : K \rightarrowtail S : k \mapsto (k, 0, 1) \text{ and}$$
$$i_2 : L \rightarrowtail S : l \mapsto (0, l, 0).$$

Suppose we are given two diaffine transformations

$$f : K \to X,$$
$$g : L \to X,$$

with factorizations $f = e^x \cdot f_0$ and $g = e^y \cdot g_0$ and scalar restriction $s : A \to B$. Define a dilinear map $h_0 : S \to X : (k, l, t) \mapsto f_0(k) + g_0(l) + s(t)(x - y)$. Then we have a diaffine transformation $h = e^y \cdot h_0$ which does the job, in fact, $h \cdot i_1 = f$, and $h \cdot i_2 = g$. Since i_2 is linear, we have $h(0) = h(i_2(0)) = g(0) = y$. Hence the affine part of h is uniquely determined. If we had two candidates h and h^* for universal arrows, they would only differ in their dilinear parts h_0 and h_0^*. But then, their difference $d = h_0 - h_0^*$ would vanish on all elements of shape $(0, l, 0)$, $l \in L$ and on all $(k, 0, 1)$, $k \in K$. The latter implies that $d(0, 0, 1) = 0$, and by dilinearity of d, $d(k, 0, 0) = d((k, 0, 1) - (0, 0, 1)) = 0$, whence the uniqueness of h.

On the other hand, if M is non-empty, so are K and L. We then have two arrows $u = i_1 \cdot f, v = i_2 \cdot g : M \rightrightarrows K \amalg L$ from the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & K \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i_1} \\ L & \xrightarrow{i_2} & K \amalg L \end{array} \tag{E.26}$$

and we are done if we can show that there is a coequalizer[3] of the couple u and v. If we have the factorizations $u = e^t \cdot u_0$ and $v = e^s \cdot v_0$, take the quotient module $E = K \amalg L / A(t - s) + \mathrm{Im}(u_0 - v_0)$. Clearly, the projection $p : K \amalg L \to E$ equalizes u and v. If $r : K \amalg L \to X$ is any diaffine transformation with scalar restriction $s : A \to B$ and equalizing the couple u and v, then r has a unique factorization through E. In fact, we may suppose without loss of generality that r is dilinear. In this case, r has the required factorization since it annihilates $A(t - s) + \mathrm{Im}(u_0 - v_0)$.

With this construction we define $K \amalg_M L = E$ and obtain a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & K \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p \cdot u} \\ L & \xrightarrow{p \cdot v} & K \amalg_M L \end{array} \tag{E.27}$$

which is the required pushout diagram in **Mod**. Observe that this proof technique—build the sum and then the coequalizer—is a special case of the fact that existence of fiber sums is equivalent to existence of sums and coequalizers provided that we have an initial object, see Appendix G.2.1. QED.

Proposition 98 *A dilinear morphism $f : M \to N$ over scalar restriction $g : A \to B$ is mono iff f is diinjective, i.e., iff f, g are both injective.*

Proof. If both, f, g are injective, then clearly the dilinear morphism is mono. If f is not injective, there are two different affine morphisms $k_i : 0_{\mathbb{Z}} \to M, i = 1, 2$, which are equalized by f. If the scalar restriction g is not injective, there are two different ring homomorphisms $r_1, r_2 : \mathbb{Z}[X] \to A$ on the polynomial ring $\mathbb{Z}[X]$ with $r_1(X) \in ker(g), i = 1, 2,$, and the zero morphism $= 0_{\mathbb{Z}[X]} \to M$ for these two scalar restrictions does the job.

E.4 Complements of Commutative Algebra

In this section, all coefficient rings are commutative.

[3] See Appendix G.2.1.

E.4.1 Localization

See also [146, II] for concepts and facts described in this section.

Let S be a multiplicative subset of a ring A, i.e., $st \in S$ for all $s, t \in S$, and $1 \in S$. The *localization* $S^{-1}A$ is the set of equivalence classes of $A \times S$ modulo the relation $(a, s) \sim (a', s')$ iff there is $t \in S$ such that $t(as' - a's) = 0$. The equivalence class of (a, s) is denoted by the fraction a/s or $\frac{a}{s}$. It is a ring by the well-defined addition $a/s + a'/s' = (as' + a's)/ss'$ and multiplication $a/s.a'/s' = aa'/ss'$. The canonical map $i_S : A \to S^{-1}A : a \mapsto a/1$ is a ring homomorphism with the universal property that for any ring homomorphism $f : A \to B$ such that $f(S) \subset B^\times$, there is a unique ring homomorphism $j : S^{-1}A \to B$ such that $f = j \circ i_S$. The ring $S^{-1}A$ is called the *localization of A in S*. Classical example: A is a domain (no zero divisors), $S = A - \{0\}$, whence $S^{-1}A$ is the classical *field $fr(A)$ of fractions over A*.

If M is an A-module, the localization $S^{-1}M$ is the set $S^{-1}M$ of equivalence classes of $M \times S$ for the equivalence relation $(m, s) \sim (m', s')$ iff there is $t \in S$ such that $t(ms' - m's) = 0$. The addition $m/s + m'/s' = (ms' + m's)/ss'$ and the scalar multiplication $r/s.m/t = rm/st$ makes $S^{-1}M$ a $S^{-1}A$-module. One has the canonical dilinear homomorphism $i_M : M \to S^{-1}M : m \mapsto m/1$ with respect to the homomorphism i_S. It has this universal property: For every homomorphism $f : M \to N$ of A-modules, such that every dilatation $s? : N \to N : n \mapsto s.n$ is bijective, there is a unique homomorphism of A-modules $j : S^{-1}M \to N$ such that $f = j \circ i_M$. It is easily seen that the tensor product $S^{-1}A \otimes_A M$, together with the canonical homomorphism of A-modules $M \to S^{-1}A \otimes_A M : m \mapsto 1 \otimes m$ is isomorphic to the localization $S^{-1}M$.

For the multiplicative set $S_s = \{1, s, s^2, s^3, \ldots\}, s \in A$, one writes A_s, M_s instead of $S_s^{-1}A, S^{-1}M$. For a prime ideal $q \subseteq A$, the complement $S = A - q$ is multiplicative by definition. And one then writes A_q, M_q instead of $S^{-1}A, S^{-1}M$.

By the universal property of localization, if $f : A \to B$ is a ring homomorphism and if $S \subseteq A, T \subseteq B$ are multiplicative sets such that $f(S) \subseteq T$, then there is a canonical ring homomorphism $f_{S,T} : S^{-1}A \to T^{-1}B$ which extends f. If $g : M \to N$ is a dilinear homomorphism over f, it extends uniquely to a dilinear homomorphism $g_{S,T} : T^{-1}M \to T^{-1}N$ over $f_{S,T}$. For a multiplicative set $S \in A$ let $S' = \{t \in A | \text{there exists } a \in A, s \in S \text{ such that } s = at\}$ be the *saturation of S*. Then the canonical homomorphisms $S^{-1}A \to s'^{-1}A$ and $S^{-1}M \to s'^{-1}M$ are isomorphisms and we may identify the two corresponding localizations. In particular, if $s \in A$, we identify A_s, M_s with $S_s'^{-1}A, S_s'^{-1}M$. So if for $s, t \in A$, one has $S_s' \subseteq S_t'$, one has canonical homomorphisms $A_s \to A_t, M_s \to M_t$.

Proposition 99 *If $i_S : A \to S^{-1}A$ is the localization homomorphism, the inverse image $q \mapsto i_S^{-1}(q)$ is an order preserving bijection from the set of maximal (resp. prime) ideals in $S^{-1}A$ to the set of maximal (resp. prime) ideals in A which are disjoint from S. In particular, if $S = A - p$ for a prime ideal p, the localization A_p is a local ring with maximal ideal $m_p = p_p$ and the residue field $\kappa_p = A_p/m_p$ is isomorphic to the field of fractions $fr(A/p)$.*

E.4.2 Projective Modules

Definition 158 *An A-module P is* projective *iff it is a direct summand of a free A-module. Equivalently, it is projective if for each pair of homomorphisms $u : P \to N$, $v : M \to N$, v an epimorphism, there is a homomorphism $w : P \to M$ such that $u = v \circ w$.*

Let $U \mapsto U^{\star\star}$ be the bidual functor on A-modules. Then a direct summand $U \subset V$ is mapped into a direct summand $U^{\star\star}$ of $V^{\star\star}$. Now, if U is projective, it is a direct summand of a free module $R^{(n)}$. It is easily seen that the bidual map $R^{(n)} \to (R^{(n)})^{\star\star}$ is injective. Therefore, if U is projective, the bidual map $U \to U^{\star\star}$ is also injective. If U is finitely generated and projective, the bidual map is an isomorphism.

Let $i : N \to M$ be the inclusion of a submodule N of an R-module M. For any $x \in M$ and positive exponent r, let

$$\wedge^r x : \bigwedge^r N \to \bigwedge^{r+1} M : y \mapsto \bigwedge^r i(y) \wedge x$$

the linear map defined by the rth exterior power of i and the wedge product with x in the exterior algebra of M. This defines a linear map

$$\wedge^r : M \to Lin_R(\bigwedge^r N, \bigwedge^{r+1} M) \tag{E.28}$$

which has the following property:

Lemma 94 *With the above notation, if N is a direct factor of M which is locally free of rank r, then $Ker(\wedge^r) = N$.*

Proof. Let $x \in M$. In the special case where N is free of rank r, if $\bigwedge^r N = R.u$ for a basis vector u of $\bigwedge^r N$, the claim $x \in Ker(\wedge^r)$ is equivalent to $u \wedge x = 0$. But this follows from [145, ch.III, §7, no.9, Prop.13]. In the general case, clearly the condition is sufficient. Conversely, suppose $\wedge^r x = 0$. Recall from [146, ch.II, §5 no.3, Th.2] that an R-module is projective of rank $r \in \mathbb{N}$ iff it is locally free of rank r. Notice that localizing commutes with exterior powers ([146, ch.II, §2 no.8]), so we may localize in $f \in R$ such that for the localized element $x_f \in M_f$, we have $\wedge^r x_f \in N_f$ and N_f is free of rank r, whence $x_f = 0$. As there is a cover of $\mathrm{Spec}(R)$ with basic open sets $D(f_i)$ associated with localizations R_{f_i} such that N_{f_i} is free, we deduce that $x \in N$. QED.

E.4.3 Injective Modules

Proposition 100 ([145, ch.II, §2, exercise 11]) *For an R-module M, the following properties are equivalent:*

(*i*) *The functor $Lin_R(?, M)$ is exact.*
(*ii*) *The functor $Lin_R(?, M)$ is exact on short exact sequences.*
(*iii*) *For every R-module E and every linear injection $F \rightarrowtail E$, every linear map $F \to M$ extends to a linear map $E \to M$.*
(*iv*) *For every ideal $\mathfrak{a} \subset R$ and every linear map $f : \mathfrak{a} \to M$, there is $m \in M$ such that $f(a) = a.m$ for all $a \in R$.*
(*v*) *M is a direct factor of every module which contains it.*
(*vi*) *For every R-module E which is sum of M and of a module I with one generator, M is a direct factor of E.*

Definition 159 *An R-module M is said to be* injective *iff it has the equivalent properties of Proposition 100. The ring R is said to* self-injective *iff it is injective as a (left) module over itself.*

Exercise 87 *Show the following statement: A direct sum of R-modules is injective iff each factor is.*

Example 108 Let $1 < N$ be a natural number. Then the Ring $\mathbb{Z}_N$ is self-injective. In fact, let $N = p_1^{n_1} \cdot \ldots p_r^{n_r}$ be its prime factorization (see Appendix D.2). Then $\mathbb{Z}_N \xrightarrow{\sim} \mathbb{Z}_{p_1^{n_1}} \times \ldots \mathbb{Z}_{p_r^{n_r}}$, and injectivity can be checked on each factor, so suppose $N = p^n$. We apply criterion (iv) of Proposition 100. An ideal $\mathfrak{a}$ in $\mathbb{Z}_{p^n}$ is generated by $p^m, m \leqslant n$, and we have the isomorphism of $\mathbb{Z}_{p^n}$-modules $\mathfrak{a} \xrightarrow{\sim} \mathbb{Z}_{p^{(n-m)}}$. Then a linear map $f : \mathbb{Z}_{p^{(n-m)}} \to \mathbb{Z}_{p^n}$ evaluates to $f(a) = f(a.1_{\mathbb{Z}_{p^{(n-m)}}}) = a.f(1_{\mathbb{Z}_{p^{(n-m)}}})$, and we are done.—In particular, every free $\mathbb{Z}_N$-module $\mathbb{Z}_N^r$ is injective.

The self-injectiveness of $\mathbb{Z}_{p^n}$ also follows from this:

Proposition 101 ([299, Proposition 21.5]) *Let R be a zero-dimensional local ring. The following are equivalent:*

(i) *R is self-injective.*
(ii) *The socle of R is simple.*

In fact, the socle of $\mathbb{Z}_{p^n}$ is isomorphic to $\mathbb{Z}_p$, the simple group of order p and it is well known that $\mathbb{Z}_{p^n}$ is zero-dimensional.

Proposition 102 *A finitely generated $\mathbb{Z}_{p^n}$-module M is injective iff it is free of finite rank.*

Proof. Clearly, by Example 108, a free module of finite rank is injective. Conversely, by the main theorem on finitely generated abelian groups E.2 we have $M \tilde{\to} \mathbb{Z}_{p^{n_1}} \times \ldots \mathbb{Z}_{p^{n_k}}, n_i \leqslant n$. By exercise 87 above, it suffices to see that $\mathbb{Z}_{p^m}$ cannot be injective if $m < n$. In fact, if the injection $\mathbb{Z}_{p^m} \leftarrow \mathbb{Z}_{p^n} : x \mapsto p^{n-m}.x$ had a left inverse h, we would have $x = h(p^{n-m}.x) = p^{n-m}.x$ which only works for $x = 0$, a contradiction. QED.

E.4.4 Lie Algebras

Definition 160 *For a module L over commutative ring R, a* Lie algebra *structure is an R-bilinear multiplication $[\ ] : L \times L \to L$, the* Lie bracket*, such that $[xx] = 0$ identically, and the* Jacobi identity

$$[x[yz]] + [y[zx]] + [z[xy]] = 0$$

holds for all $x, y, z \in L$. A homomorphism of Lie algebras $f : L_1 \to L_2$ is a linear homomorphism such that $f([xy]) = [f(x)f(y)]$ for all $x, y \in L_1$. The corresponding category of Lie algebras over R is denoted by $\mathbf{Lie}_R$.

Example 109 If $L \tilde{\to} R^n$ is free, and if (x_i) is a basis, a Lie algebra structure on L is defined by bilinearity and skew symmetry (which follows from $[xx] = 0$) of the Lie bracket if the Lie brackets $\sum_k a_{ij}^k x_k = [x_i, x_j], i < j$ are known. The condition for such a bracket to generate a Lie algebra is

$$a_{ii}^k = 0, \text{ all } i, k, \tag{E.29}$$

$$a_{ij}^k + a_{ji}^k = 0, \text{ all } i < j, k, \tag{E.30}$$

$$\sum_k a_{ij}^k a_{kl}^m + a_{jl}^k a_{ki}^m + a_{li}^k a_{kj}^m = 0, \text{ all } i, j, l, m. \tag{E.31}$$

The coefficients a_{ij}^k are called *structural constants of the Lie algebra in the given basis.*

Example 110 For any module L, the R-algebra of linear endomorphisms $End(L)$ becomes the *general linear algebra* $\mathfrak{gl}(L)$ by the bracket $[xy] = x \circ y - y \circ x$. A sub-Lie-algebra of a general linear algebra is called a *linear Lie algebra*. If $L \tilde{\to} R^n$ is free of rank n, we also write $\mathfrak{gl}(L) = \mathfrak{gl}(n, R)$. Its subalgebra of endomorphisms with vanishing trace (check that it is a sub-Lie-algebra!) is called the *special linear algebra* and denoted by $\mathfrak{sl}(L)$ or $\mathfrak{sl}(n, R)$.

Example 111 *Let L be any R-module with a bilinear product $x \cdot y$ (no other conditions required). A derivation is a linear endomorphism $D : L \to L$ such that $D(x \cdot y) = x \cdot D(y) + D(x) \cdot y$. The set $Der(L)$ is a submodule of $End(L)$, and in fact a Lie subalgebra of the general linear algebra $\mathfrak{gl}(L)$.*

In particular, if we take the Lie algebra structure on L, $Der(L)$ is another Lie algebra. Observe that for $x \in L$, the left multiplication $ad(x) = [x?] : y \mapsto [xy]$ is a derivation by the Jacobi identity. One has the Lie algebra homomorphism of adjunction

$$ad : L \to Der(L), \tag{E.32}$$

a representation of L in the general linear algebra of L. A derivation $ad(x)$ is called inner*, any other is called* outer *derivation.*

Proposition 103 *If x is a nilpotent endomorphism in a linear algebra, then its adjoint $ad(x)$ is also a nilpotent endomorphism.*

See [491, p.12] for the easy proof.

Suppose that L is a linear algebra in $End(V)$ for an R-module V, and that the conjugation $Int_e, e \in GL(V)$ leaves L invariant. Then evidently the conjugation is an automorphism of L. Suppose now that R is a $\mathbb{Q}$-algebra. If $ad(x)$ is nilpotent, then the exponential

$$exp(ad(x)) = 1 + ad(x) + ad(x)^2/2! + \dots ad(x)^k/k! + \dots \tag{E.33}$$

is defined. We have:

Lemma 95 *If $ad(x)$ is nilpotent, then $exp(ad(x))$ is an automorphism of the Lie algebra L. Moreover, if x is nilpotent, $exp(x)$ is defined and we have*

$$Int_{exp(x)} = exp(ad(x)).$$

See [491, p.9] for the proof.

F

Algebraic Geometry

For this chapter, we refer to [146, 259, 395, 396, 300].

F.1 Locally Ringed Spaces

Given a topological space X, its system of open sets $Open_X$ is viewed as a category with inclusions as morphisms. If $f : X \to Y$ is a continuous map, the inverse image map $U \mapsto f^{-1}U$ defines a functor $Open_f : Open_Y \to Open_X$. This defines a functor $Open_? : \mathbf{Top} \to \mathbf{Cat}$ into the category of categories and functors[1].

Let $\mathbf{C}$ be a category of sets with some additional algebraic structure, such as the categories

$$\mathbf{Mod}, \mathbf{Mon}, \mathbf{Grp}, \mathbf{Ab}, \mathbf{Rings}, \mathbf{ComRings}$$

of modules, monoids, groups, abelian groups, rings, or commutative rings, respectively. A contravariant functor $F : Open_X \to \mathbf{C}$ is called a $\mathbf{C}$-space (this is a presheaf plus the algebraic morphism conditions). For example, a ringed space is just a **Rings**-space. In the present context, we always suppose that a ringed space is one with values in **ComRings**, i.e., a commutatively ringed space. The set of $\mathbf{C}$-spaces on X is denoted by $\mathbf{C}_X^{spaces}$. The contravariant functor $Open_?$ induces a functor $\mathbf{C}_?^{spaces} : X \mapsto \mathbf{C}_X^{spaces}$. It maps the continuous map $f : X \to Y$ to the set map $\mathbf{C}_f^{spaces} : F \mapsto F \circ Open_f$. The image $F \circ Open_f$ is denoted by f_*F and is called the *direct image of* F. If $F : Open_X \to \mathbf{C}, G : Open_Y \to \mathbf{C}$ are $\mathbf{C}$-spaces and $f : X \to Y$ is continuous, then an *f-morphism* $h : F \to G$ is a natural transformation $h : G \to f_*F$. These morphisms define an evident category, the *category* $\mathbf{C}^{spaces}$ *of* $\mathbf{C}$*-spaces*.

Suppose that the category $\mathbf{C}$ has colimits for filtered diagrams[2], such as

$$\mathbf{Rings}, \mathbf{ComRings}, \mathbf{Mod}, \mathbf{Grp}, \mathbf{Ab},$$

and take a $\mathbf{C}$-space F. For each point $x \in X$, the filtered system $Open_{X,x}$ of open neighborhoods of x defines an object $F_x = colim_{U \in Open_{X,x}} F(U)$, the *stalk of* F *at* x. So for a (commutatively) ringed space, this is a (commutative) ring. Let $h : F \to G$ be an f-morphism for $f : X \to Y$. For $x \in X$, we have the restriction $Open_{f,x} : Open_{Y,f(x)} \to Open_{X,x}$ of $Open_f$ to the neighborhood systems $Open_{Y,f(x)}$ and $Open_{X,x}$. This induces a $\mathbf{C}$-morphism $h_x : G_{f(x)} \to F_x$.

The subcategory **LocRgSpaces** of **ComRings**-spaces consists of all ringed spaces F which have a local ring F_x with maximal ideal m_x in each point $x \in X$, and of those morphisms h which induce local morphisms h_x in all stalks, i.e., $h_x^{-1}(m_x) = m_y$. It is called the *category of locally ringed spaces*. The residue

[1] Restricted to a universe, if the limitless collection bothers the reader, or even the category of partially ordered sets with order preserving maps, to stick to reality.

[2] Meaning that for any two objects in the diagram quiver, there are two arrows with these domains targeting at a common codomain.

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G. Mazzola, *The Topos of Music IV: Roots*, Computational Music Science,
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field in a point x of such a space F is the field $\kappa(x) = F_x/m_x$. For a section $s \in F(U)$ over the open set U, we denote by $s(x)$ the canonical image of s in $\kappa(x)$.

For the category **ComMod** of modules over commutative rings and dilinear homomorphisms, a **ComMod**-space F can also be described by the underlying ringed space R and the abelian group space F (same notation), together with a scalar multiplication $R(U) \times F(U) \to F(U)$ in each open set U, and the evident dilinear transition maps. One therefore also says that F *is an R-module*. Same wording, mutatis mutandis, for an R-algebra or for an R-ideal.

If we are given a **C**-space F on X which is a sheaf, and if $\mathfrak{B}$ is a topological base for X, the restriction $F|\mathfrak{B}$ to this subcategory of $Open_X$ completely determines F. If $U \in Open_X$, $U = colim(B \in \mathfrak{B}|B \subseteq U)$, and

$$F(U) = lim(F(B), B \in \mathfrak{B}).$$

Conversely, if we are given a contravariant functor $F : \mathfrak{B} \to \mathbf{C}$, we obtain a **C**-space F' by

$$F'(U) = lim(F(B), B \in \mathfrak{B})$$

and by the universally given transition morphisms. This presheaf is a sheaf if F is a *sheaf on* $\mathfrak{B}$, i.e., if for every covering (B_i) of $B \in \mathfrak{B}$ by elements of the base, the canonical application $F(B) \to \prod_i F(B_i)$ is a bijection $x \mapsto (x|B_i)$ onto the tuples (x_i) such that for every i, j and base element $B' \subseteq B_i \cap B_j$, we have $x_i|B' = x_j|B'$.

F.2 Spectra of Commutative Rings

Definition 161 *The (prime) spectrum is a contravariant functor*

$$Spec : \mathbf{ComRings} \to \mathbf{LocRgSpaces}$$

which is defined as follows: Let A, B be commutative rings, and let $f : A \to B$ be a ring homomorphism.

1. *The topological space consists of the set $Spec(A) = \{p \text{ a prime ideal in } A\}$. The closed sets are the sets of the form $V(E) = \{p|E \subseteq p\}$ for a subset $E \subseteq A$. Equivalently, a base of open sets is given by the system $D_f = \{p|f \notin p\}, f \in A$, and we have $D_f \cap D_g = D_{fg}$.*
2. *For the base $\mathfrak{D} = \{D_f|f \in A\}$, we have a sheaf on $\mathfrak{D}$, which is defined by $D_f \mapsto A_f$, the localization at the saturated multiplicative set $S(f)$ defined by f, a well-defined setup since $D_f = D_g$ iff $S(f) = S(g)$ see* [395, I.1.3.2]. *This presheaf is a sheaf on $\mathfrak{D}$, and the associated sheaf on $Spec(A)$ is denoted by $\tilde{A}$ and called the* ring sheaf associated with *A. If $p \in Spec(A)$ is a prime ideal, we have $(\tilde{A})_p \xrightarrow{\sim} A_p$, i.e., $\tilde{A} \in$ **LocRgSpaces**.*
3. *For the homomorphism $f : A \to B$, the inverse image map on prime ideals $Spec(f) : Spec(B) \to Spec(A) : p \mapsto f^{-1}p$ is defined, and we have $Spec(f)^{-1}(D_g) = D_{f(g)}$, $Spec(f)^{-1}(V(E)) = V(f(E))$, i.e., $Spec(f)$ is continuous. Furthermore, we have a canonical map $f_g : A_g \to B_{f(g)}$ which is natural and therefore induces a morphism $Spec(f) : \tilde{A} \to \tilde{B}$ over the continuous (synonymous) map $Spec(f)$. The stalk homomorphism $f_p : \tilde{A}_p \to \tilde{B}_{Spec(f)(p)}$, colimit of the natural homomorphisms $f_g : A_g \to B_{f(g)}$, is local. One therefore has a contravariant functor Spec as announced, and one often denotes $Spec(A)$, when meaning the locally ringed space $\tilde{A}$ over $Spec(A)$.*

Theorem 64 *The functor Spec is fully faithful, the inverse* global section functor *Γ of a* **LocRgSpaces***-morphism $u : \tilde{B} \to \tilde{A}$ is given by the ring homomorphism $u(Spec(A)) : A = \tilde{A}(Spec(A)) \to \tilde{B}(Spec(B)) = B$.*

See [395, I.1.6.3] for a proof. Since one often writes $F(U) = \Gamma(U, F)$ and calls the elements section above U, the theorem's notation is justified by the global section notation $\Gamma(\tilde{A}) = \Gamma(Spec(A), \tilde{A})$. Let **Aff** be the full subcategory of **LocRgSpaces** consisting of the objects which are isomorphic to prime spectra. These spaces are called *affine schemes*. We therefore have that the map

$$Spec : \mathbf{ComRings} \to \mathbf{Aff}$$

is an equivalence of categories.

If M is an A-module, we have a $\tilde{A}$-module $\tilde{M}$, whose sections on the base $\mathfrak{D}$ are defined by $\Gamma(D_g, \tilde{M}) = A_g \otimes M = M_g$, the localization of M at the multiplicative set $S(g)$.

Proposition 104 *The map $M \mapsto \tilde{M}$ is an exact*[3] *and fully faithful functor from the category of A-modules $\mathbf{Mod}_A$ to the category $\mathbf{Mod}_{\tilde{A}}$ of $\tilde{A}$-modules. It also commutes with colimits of modules, with tensor products, Hom-modules, with sums and intersections of submodules.*

The inverse to this functor is the global section functor $\tilde{M} \mapsto \Gamma(Spec(A), \tilde{M})$, which is also exact.

See [395, I.1.3] for a proof. The modules in $\mathbf{Mod}_{\tilde{A}}$, which are hit by this tilding process are the quasi-coherent ones: A module $\mathcal{M}$ over a ringed space $\mathcal{A}$ over a topological space X is *quasi-coherent* iff there is a covering X_i of X such that each restriction $\mathcal{M}_i = \mathcal{M}|X_i$ is the cokernel of a homomorphism $f_i : \mathcal{A}_i^{I_i} \to \mathcal{A}_i^{J_i}$, where $\mathcal{A}_i = \mathcal{A}|X_i$.

Theorem 65 *An $\tilde{A}$-module $\mathcal{M}$ is isomorphic to a module $\tilde{M}$ iff it is quasi-coherent.*

See [395, I.1.4.1] for a proof. This means that we have an equivalence of categories of quasi-coherent modules over $\tilde{A}$ and $\mathbf{Mod}_A$ (with linear homomorphisms).

For a ring element $f \in A$, one has $\mathrm{Spec}(A_f) \xrightarrow{\sim} D_f$. When restricting the associated ring and module sheaves $\tilde{A}, \tilde{M}$ to basic open sets D_f, this yields ring and module sheaves which are isomorphic to $\tilde{A_f}, \tilde{M_f}$.

Theorem 66 *Let M be an A-module. The following conditions are equivalent:*

(*i*) *M is projective*[4] *and finitely generated.*
(*ii*) *There is a finite family (f_i) of elements of A which generate the ideal A, i.e., $Spec(A) = \bigcup_i D_{f_i}$ such that the localizations $M_{f_i} = \Gamma(D_{f_i}\tilde{M})$ are free of finite rank over A_{f_i}.*

This is why one can also define a finitely generated projective module as being a locally free module of (locally defined) finite ranks. If the locally constant rank is constant n, the module is said to be *locally free of rank n*.

F.2.1 Sober Spaces

A topological space X is *irreducible* iff every non-empty open subset is dense, or, equivalently, if any two non-empty open sets have a non-empty intersection. A subset of a topological space is called irreducible if it is so with its relative topology. A point x of an irreducible space X is said to be *generic* iff its (always irreducible) closure $\overline{\{x\}}$ is X. We say that *a point x dominates a point y, in signs $x > y$*, iff $\overline{\{y\}} \subseteq \overline{\{x\}}$. This is a partial order relation on X. An irreducible component of a space X is a maximal irreducible subset.

Sorite 16 *These are the sorite properties concerning irreducibility:*

(*i*) *A subset of a topological space is irreducible iff its closure is.*
(*ii*) *Irreducible components are closed.*
(*iii*) *Every irreducible subset is contained in an irreducible component, in particular, a topological space is the union of its irreducible components.*
(*iv*) *The image $f(E)$ of an irreducible subset $E \subseteq X$ under a continuous map $f : X \to Y$ is irreducible.*

[3] A sequence $K \xrightarrow{f} M \xrightarrow{g} L$ of linear homomorphisms of modules is exact in M iff $Im(f) = Ker(g)$. Such exact sequences are preserved by the functor.
[4] See definition 158 in Appendix E.4.2.

Definition 162 *A topological space X is* sober *iff each closed irreducible subset has a unique generic point. Call* **Sob** *the full subcategory of the category* **Top** *of topological spaces consisting of sober spaces.*

If A is a commutative ring, and if $E \subseteq Spec(A)$, then we denote $J(E) = \bigcap_{p\in E} p$, and $\overline{E} = V(J(E))$. This ideal is prime iff E is irreducible. In this case, $\overline{E} = \overline{\{J(E)\}}$. In fact, for two points p, q in $Spec(A)$, $p > q$ iff $p \subseteq q$. In particular, $Spec(A)$ *is a sober space.* Its irreducible components correspond to the minimal prime ideals.

Proposition 105 *The canonical injection*

$$j : \mathbf{Sob} \to \mathbf{Top}$$

has a left adjoint $?^s : \mathbf{Top} \to \mathbf{Sob}$.

Proof idea. This adjoint associates with any X a sober space X^s which is defined as follows. Its points are the irreducible closed sets in X. The open sets are the sets $V^s = \{Y \in X^s | Y \cap V \neq \varnothing\}$, where V varies over all open sets in X. Clearly, $(V \cap W)^s = V^s \cap W^s$ and $(\bigcup W_i)^s = \bigcup W_i^s$ for any family (W_i) of open sets.

On continuous maps $f : X \to Y$, the functor acts via $f^s : X^s \to Y^s : E \mapsto \overline{f(E)}$. One has a canonical continuous map $q_X : X \to X^s : x \mapsto \overline{\{x\}}$ and a commutative diagram of continuous maps:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_Y} \\ X^s & \xrightarrow{f^s} & Y^s \end{array} \tag{F.1}$$

The map $q_X : X \to X^s$ is a homeomorphism if X is sober. The adjunction is given by the mutually reciprocal maps $\mathbf{Top}(X, j(Y)) \to \mathbf{Sob}(X^s, Y) : f \mapsto q_Y^{-1} \circ f^s$ and $\mathbf{Sob}(X^s, Y) \to \mathbf{Top}(X, j(Y)) : g \mapsto g \circ q_X$.

Lemma 96 *The canonical continuous map $q_X : X \to X^s$ is a quasi-homeomorphism,. i.e., the inverse image map $2^{X^s} \to 2^X$ is a bijection between the open sets of X^s and those of X.*

F.3 Schemes and Functors

A scheme $(X, \mathcal{O}_X)$ is a ringed space $\mathcal{O}_X$ on X which locally is isomorphic to a spectrum of a commutative ring, i.e., there is an open covering (X_i) of X and a family A_i of rings such that $(X_i, \mathcal{O}_X|X_i) \xrightarrow{\sim} Spec(A_i)$. The category **Schémas** of schemes is the subcategory of **LocRgSpaces** whose objects are schemes. By Yoneda, we have a fully faithful functor $Y : \mathbf{Schémas} \to \mathbf{Schémas}^{@}$.

Proposition 106 *The restriction*

$$Y_{\mathbf{Aff}} : \mathbf{Schémas} \to \mathbf{Aff}^{@}$$

is fully faithful. Equivalently, the corresponding functor

$$Y_{\mathbf{ComRings}} : \ \mathbf{Schémas} \to \mathbf{ComRings}_{@}$$

into the category $\mathbf{ComRings}_{@}$ *of covariant set-valued functors on* **ComRings** *is fully faithful.*

This means that we may consider schemes as special covariant functors on the category of commutative rings. The functors which correspond to schemes are characterized by a sheaf condition:

Property 3 *We are given a functor $G \in \mathbf{ComRings}_{@}$. For every ring $A \in \mathbf{ComRings}$, and every finite family (f_i) of elements of A which generate A as an ideal, the diagram*

$$G(A) \to \prod_i G(A_{f_i}) \rightrightarrows \prod_{i,j} G(A_{f_i f_j})$$

is exact, we say (by abuse of language, but theoretically justifiable) that G is a sheaf.

Then the full subcategory of $\mathbf{ComRings}_{@}$ consisting of sheaves G in the sense of property 3, together with the property that there is a family of rings R_t and morphisms[5] $a_t : R_t@ \to G$ such that for every field K, $G(K) = \bigcup_t R_t@K$, comprises the functors which are isomorphic to images of schemes under the Yoneda map $Y_{\mathbf{ComRings}}$, see [395, I.2.3.6] and [300, VI.2] for details. This means that schemes are characterized without any reference to the geometry of ringed spaces. See also [639, III.3] for the relation of this setup to the systematic topos-theoretic restatement of the schemes in terms of the *Zariski site*.

The most important universal property of the category of schemes is that it has fiber products. In the affine case, we have $Spec(A) \times_{Spec(C)} Spec(B) \xrightarrow{\sim} Spec(A \otimes_C B)$.

F.4 Algebraic and Geometric Structures on Schemes

If a scheme is viewed as a set-valued functor on rings, the sets may also be enriched by algebraic structures, such as groups, monoids, etc., to yield a category $\mathbf{C}$. We then view a scheme as a functor $S : \mathbf{ComRings} \to \mathbf{C}$, and say that S is a $\mathbf{C}$-scheme, for example, an abelian group-scheme if $\mathbf{C} = \mathbf{Ab}$.

Example 112 For $0 \leqslant n$, we have the additive group scheme $\mathbb{A}^n$ whose functor is $\mathbb{A}^n(R) = R^n$, with the canonical addition of this free module, and the canonical transitions $\mathbb{A}^n(R) \to \mathbb{A}^n(S)$ for a ring homomorphism $f : R \to S$.

Example 113 The n-dimensional linear group scheme is given by the functor $R \mapsto GL(n,R) \subset \mathbb{M}_{n,n}(R) \xrightarrow{\sim} \mathbb{A}^{n^2}(R)$, together with the canonical map $GL(n,R) \to GL(n,S)$ for a ring homomorphism $f : R \to S$. The set $GL(n,R)$ is defined as the set of $n \times n$-matrices M with invertible determinant: $det(M) \in R^\times$. The functor is represented by the affine scheme $GL_n = Spec(\mathbb{Z}[X_{ij}, 1 \leqslant i,j \leqslant n]_{det})$, where $det = Det(X_{ij})$. The group structure is the multiplication of invertible matrices.

F.4.1 The Zariski Tangent Space

For a field K, we are given a K-scheme X, i.e., a scheme $s : X \to Spec(K)$ in the comma category $\mathbf{Schémas}/Spec(K)$ (see Section G.2.1). Suppose that we have a K-rational point $x : Spec(K) \to X$, i.e., a section of s. This means that the corresponding K-algebra $\mathcal{O}_{X,x}$ has an isomorphism $K \xrightarrow{\sim} \kappa_{X,x} = \mathcal{O}_{X,x}/m_{X,x}$. The *Zariski tangent space in x* is the linear K-dual $T_{X,x} = (m_{X,x}/m_{X,x}^2)^*$. Consider the K-scheme $D_K = Spec(K[\varepsilon])$ over the dual numbers $K[\varepsilon]$, see Example 106 in Appendix D.1.1. It has the K-rational point $\varepsilon : Spec(K) \to D_K$ corresponding to the projection $K[\varepsilon] \to K : \varepsilon \mapsto 0$.

Proposition 107 *With the above hypotheses and notation, there is a bijection of the Zariski elements t of the tangent space $T_{X,x}$ and the morphisms $\tau : D_K \to X$ of K-schemes which map the K-rational point ϵ to the K-rational point x.*

See [300, VI.1.3] for a proof. In particular, if the scheme X is given by its functor on rings, this means that the tangents are special elements of the evaluation of the functor in dual numbers $X(K[\varepsilon])$. For example, if $X = \mathbb{A}^1_K = \mathbb{A}^1 \times_{\mathbb{Z}} K$, we have tangents $x + \varepsilon.\tau, \tau \in K$, over the rational point $x \in K \xrightarrow{\sim} \mathbb{A}^1_K(K)$.

[5] $R_t@$ is the covariant functor on rings, i.e., $R_t@R = Hom_{\mathbf{ComRings}}(R_t, R)$.

F.5 Grassmannians

A *subfunctor* $G \to F$ *in* $\mathbf{ComRings}_@$ *is open* iff for every morphism $a : R@ \to F$ (corresponding to an element $a \in F(R)$ via Yoneda), the fiber product projection $G \times_a R@ \to R@$ is isomorphic to the functor of an open subscheme of $Spec(R)$. Clearly, then, if $b : X@ \to F$ is a morphism from a representable functor $X@$ of a scheme X, then the projection $G \times_b X@ \to X@$ is isomorphic to an open subscheme of X. An *open covering of a functor* F in $\mathbf{ComRings}_@$ is a family $(g_i : G_i \to F)$ of open subfunctors of F such that the fiber product projections $G_i \times_b X@ \to X@$ for morphisms $b : X@ \to F$ from the representable functor of a scheme X define an open covering of X. For example, the open subfunctors of an affine scheme $Spec(R)$ are the functors $F_I : \mathbf{ComRings} \to \mathbf{Ens}$ of form $F_I(S) = \{f : R \to S | f(I)S = S\}$, where I is an ideal in R.

The Grassmann scheme $Grass_{r,n}$ is defined for any couple $0 \leqslant r \leqslant n$ of natural numbers by the functor

$$Grass_{r,n}(R) = \{V \subseteq R^n | R^n/V \text{ locally free of rank } r\},$$

which for a ring homomorphism $R \to S$ maps the exact sequence

$$0 \to V \to R^n \to R^n/V \to 0$$

to the exact sequence

$$0 \to Im(S \otimes_R V) \to S^n \to S \otimes_R (R^n/V) \to 0,$$

where the image of the tensorized space $S \otimes_R V$ is the image of V under this map. The locally free quotient remains locally free since the localization on R carries over to a localization over S: For $f \in R$, and its image $f' \in S$, we have $(S \otimes_R R^n/V)_{f'} \xrightarrow{\sim} S_{f'} \otimes_{R_f} (R^n/V)_f$.

The functor $Grass_{r,n}$ is covered by the following open subfunctors. Let $i_\cdot = i_1, i_2, \ldots i_r$ be an increasing subsequence $1 \leqslant i_1 < i_2 < \ldots i_r \leqslant n$. We have the affine open subfunctors $Grass_{n,r,i_\cdot}(R)$ of those submodules $V \subset R^n$ such that the factor $R^{i_\cdot} = \bigoplus_{j=1,\ldots r} R.e_{i_j}$ of R^n projects isomorphically onto the quotient R^n/V. If $i'_\cdot$ denotes the complementary increasing sequence, $Grass_{n,r,i_\cdot}(R)$ identifies to the set of graphs Γf of linear maps in $Lin_R(R^{i'_\cdot}, R^{i_\cdot})$, i.e., to $n \times (n-r)$-matrices with columns $(e_{i'_k}, f(e_{i'_k}))^t$. In fact, the isomorphism $R^{i_\cdot} \xrightarrow{\sim} R^n/V$ corresponds to an isomorphism $V \xrightarrow{\sim} R^{i'_\cdot}$, and this makes V a graph of a linear map in $Lin_R(R^{i'_\cdot}, R^{i_\cdot})$. The fact that these open subfunctors (represented by affine schemes $\mathbb{A}^{r\times(n-r)}$) cover the Grassmannian results from the situation over a field, where the covering is evident.

Proposition 108 *Let n be a positive natural number. Then the subfunctor*

$$B_n : R \mapsto \{x \in R^n |\ x \text{ is part of a basis of } R^n\} \tag{F.2}$$

of the affine n-space $\mathbb{A}^n$ over $\mathbb{Z}$ is an open subscheme.

Proof. Consider the open subscheme GL_n of $\mathbb{A}^{n^2}$. Then B_n is the image of GL_n under the projection onto the first column (n_{1j}) which by [396, IV/2, 2.4.7] is (universally) open. QED.

Lemma 97 *If X is an S-scheme, E a quasi-coherent O_X-module, then any section $s : S \to Grass_r(E)$ is a closed immersion.*

Proof. By [395, Proposition (9.7.7)], $Grass_r(E)$ separated over S, and by [395, Corollaire (5.2.4)], a section of such a structural morphism is a closed immersion. QED.

Lemma 98 *If R is a product of local rings of finite length, then for two elements x, y of an R-module M, $R^\times x = R^\times y$ iff these elements generate the same space, i.e., $R.x = R.y$.*

Without loss of generality, we may suppose that R is local with maximal nilpotent ideal m. Clearly, the condition is sufficient. Suppose now that $R^\times x \neq R^\times y$, and therefore $R^\times x \cap R^\times y = \varnothing$. Then, $R.x = R.x$ implies $R^\times x \subset m.y$, since $R = R^\times \cup m$. But then we have $x \in m.y$, and symmetrically $y \in m.x$, which gives $x \in m^k.y$ for all powers k, and m being nilpotent yields $x = y = 0$, a contradiction. QED.

F.6 Quotients

If G is a finite group, and if $(X, \mathcal{O}_X)$ is a scheme, a group action of G on X can be given by a group homomorphism $\alpha : G \to Aut(X)$. This can also be seen as a morphism of schemes $\alpha' : G_{\mathbb{Z}} \times_{Spec(\mathbb{Z})} X \to X$ with the functorially described axioms of group actions associated with the functors of the schemes $G_{\mathbb{Z}}, X$, the scheme $G_{\mathbb{Z}} = Spec(\mathbb{Z}^G)$ is a group scheme whose multiplication is associated with the group multiplication $\mu : G \times G \to G$ via the ring homomorphism $\mu' : \mathbb{Z}^G \to \mathbb{Z}^{G\times G} \tilde{\to} \mathbb{Z}^G \otimes_{\mathbb{Z}} \mathbb{Z}^G$. The scheme $G_{\mathbb{Z}}$ is finite and locally free over $\mathbb{Z}$. The *set-theoretic orbits* of the action α are the equivalence classes defined by the relation on the product set $X \times X$, image of the set map $G \times X \to X \times X : (g, x) \mapsto (\alpha(g).x, x)$. If we use the schema-theoretic map $\alpha' : G_{\mathbb{Z}} \times_{Spec(\mathbb{Z})} X \to X$, the cokernel functor of the pair $pr_2, \alpha' : G_{\mathbb{Z}} \times_{Spec(\mathbb{Z})} X \rightrightarrows X$ of functor morphisms, if it exists, is called the *scheme functor of orbits of X under the action of G*. We have this particular case of [259, III,2.6.1]:

Theorem 67 *With the above notation, if G is a finite group and $\alpha' : G_{\mathbb{Z}} \times_{Spec(\mathbb{Z})} X \to X$ the group action associated with an 'abstract' action $\alpha : G \to Aut(X)$ on the scheme X, such that every set-theoretic orbit is contained in an affine open subscheme of X, then there is a scheme-functor of orbits $Y = coker(pr_2, \alpha')$ and the associated diagram of schemes (qua locally ringed spaces)*

$$G \times_{Spec(\mathbb{Z})} X \rightrightarrows X \to Y$$

is exact.

G

Categories, Topoi, and Logic

For a comprehensive introduction to category theory, see [637]. For topos theory and sheaves see [639], for topos theory and logic, see [376].

G.1 Categories Instead of Sets

One may rebuild mathematics from categories rather than from sets. In this framework, the most radical approach is the arrow-only definition of a category[1]:

Definition 163 *A category* $\mathbf{C}$ *is a collection of objects* $f, g, h, \ldots$ *which are called* morphisms*, together with a partial composition* $f \circ g$ *which yields morphisms of* $\mathbf{C}$*. An identity is a morphism* e *such that, whenever defined, we have* $e \circ f = f$ *and* $g \circ e = g$*. We have these axioms:*

1. *Whenever one of the two compositions* $(f \circ g) \circ h, f \circ (g \circ h)$ *is defined, both are defined and they are equal; we denote the resulting morphism by* $f \circ g \circ h$*.*
2. *If* $f \circ g, g \circ h$ *are both defined,* $(f \circ g) \circ h$ *is defined.*
3. *For every morphism* f *there are two identities, a 'left' identity* e_L *and a 'right' identity* e_R*, such that* $e_L \circ f, f \circ e_R$ *are defined (and necessarily equal to* f*).*

It is easily seen that two right (left) identities of a morphism f are necessarily equal; they are called the *domain of* f (*codomain of* f) and are denoted by $dom(f)$ ($codom(f)$). To make domain and codomain evident, one also writes $f : a \to b$ with $a = dom(f), b = codom(f)$ instead of f. For two morphisms a, b, the collection of those f with $dom(f) = a, codom(f) = b$ is denoted by $Hom(a, b), Hom_{\mathbf{C}}(a, b), \mathbf{C}(a, b), \ldots$ according to the specific situation. Evidently, no morphism can be a member of $Hom(a, b)$ and of $Hom(a', b')$ if either $a \neq a'$ or $b \neq b'$, i.e., the Hom collections form a partition of $\mathbf{C}$ (in the non-set-theoretic common sense).

Exercise 88 *Two identities* e, e' *of* $\mathbf{C}$ *can be composed iff they are equal, and then* $e \circ e = e$ *(identities are idempotent).*

In a more conservative understanding of categories, the identities are associated with the "objects" of a category, which are a second type of concepts, but do not enrich the category except in the way it is constructed. The identification of objects and identities is carried out as often as possible in our present text.

[1] Mac Lane calls this type of set-less categories "metacategories", and reserves the proper term "category" for metacategories which are built upon sets. We do however preconize the foundational character of metacategories and therefore omit the "meta" prefix. However, we then should provide a germ for existing categories, in order to get off ground as with axiomatic set theory. See [639, VI.10] for a discussion of the foundation of mathematics via topoi.

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In either case, the collection of identities (qua objects) is denoted by ${}_0\mathbf{C}$ or $Ob(\mathbf{C})$, whereas the morphisms are denoted by ${}_1\mathbf{C}$ or $Mor(\mathbf{C})$. To stress the morphic character of an identity e (in contrast to the underlying object in the conservative understanding), one also writes Id_e instead of e.

In a category, a morphism f is *mono, a monomorphism*, iff for any two compositions $f \circ g, f \circ g'$, the equality $f \circ g = f \circ g'$ implies $g = g'$. The morphism f is *epi, an epimorphism*, iff for any two compositions $g \circ f, g' \circ f$, the equality $g \circ f = g' \circ f$ implies $g = g'$. The morphism f is called a *section* if there is a left inverse g, i.e., $g \circ f = dom(f)$; f is called a *retraction* if it has a right inverse h, i.e., $f \circ h = dom(h)$. A morphism f that is a section and a retraction is *iso, an isomorphism*. If $dom(f) = codom(f)$, the morphism is called *endo, an endomorphism*. An endomorphism which is an isomorphism is called *auto, an automorphism*. The collection of endomorphisms for a domain c is denoted by $End(c)$, whereas the collection of automorphisms for c is denoted by $Aut(c)$. If these collections are sets, they define monoids $End(c)$ and groups $Aut(c)$ with the identity Id_c as unit.

Exercise 89 *The composition of two monomorphisms, epimorphisms, isomorphisms, endomorphisms, and automorphisms, if defined, shares, each of these properties.*

G.1.1 Examples

Example 114 The category **Ens** of all sets. The morphisms are the set maps between existing sets, and the composition is the usual composition of set maps.

Remark 33 *Usually, the delicate comprehension axiom which can cause contradictory constructions of sets, is avoided by a strong restriction of the available sets. One takes a very large set U, which has the properties of a "universe", i.e., it is stable in the following sense:*

- *If $x \in U$, then $x \subset U$;*
- *If $x, y \in U$, then $\{x, y\} \in U$;*
- *If $x \in U$, then $2^x \in U$ (the set of all subsets, the powerset);*
- *A set of all natural[2] numbers $\mathbb{N}$ is element of U;*
- *If $f : x \to y$ is a surjective function with $x \in U, y \subset U$, then $y \in U$*

One then restricts the **Ens** *objects to the elements of the universe U and says that these are* small sets. *We denote such a category of small sets by* $\mathbf{Ens}_U$.

Example 115 Given a quiver $Q = (\text{head}, \text{tail} : A \rightrightarrows V)$ (see Section C.2.2), the *path category* $P(Q)$ has the paths as morphisms, the identities are the lazy paths, and the composition is the path composition. Here, the vertexes are separate concepts which can be identified (and in fact *are* identified in our construction) with the lazy paths. All paths are mono and epi, but only the identities are isomorphisms. The terminology "quiver" stems from algebra, in category theory, a quiver is more known as a "diagram scheme".

Relations among paths give rise to *quotient categories* as follows: Suppose that we are given any binary relation $\sim$ between some paths of equal domain and codomain. Consider the smallest equivalence relation $\sim'$ among paths which contains $\sim$ and is a 'two-sided ideal' in the sense that for $f \sim' g$ with $dom(f) = dom(g) = d, codom(f) = codom(g) = c$ and h, k with $dom(h) = c, codom(k) = d$, we have $c \circ f \sim' c \circ g$ and $f \circ d \sim' g \circ d$. Then we obtain a new category, the quotient category $P(Q)/\sim$, and its morphisms are the equivalence classes of paths, while the composition is the composition of representatives of these classes. In the language of category theory, the relation $\sim$ is called a *commutativity relation* of the given diagram scheme.

Example 116 Fix a ring R, the *matrix category over* R is the collection $\mathbb{M}_R$ of all $m \times n$-matrices $M = (m_{i,j})$ with coefficients in R and for any row and column numbers m, n, together with the usual matrix multiplication $M \cdot N$ as composition. The identities are all the identity matrices $E_n, n = 1, 2, \ldots$ (over R). We evidently

[2] For example the set of finite ordinals $0 = \varnothing, 1 = \{0\}, 2 = \{0, 1\}, \ldots n, n^+ = n \cup \{n\}, \ldots$

have $Hom_{\mathbb{M}_R}(E_n, E_m) = \mathbb{M}_{m,n}(R)$. In particular, the vectors in R^n are identified with the morphisms in $Hom_{\mathbb{M}_R}(E_1, E_m) = \mathbb{M}_{1,n}(R)$.

Example 117 Given a category $\mathbf{C}$, the isomorphism classes of $\mathbf{C}$-objects define a *skeleton* category $\mathbf{C}/iso$: For each isomorphism class, select a representative and then consider the full subcategory[3] of $\mathbf{C}$ on these representative objects. Clearly $\mathbf{C}/iso$ is defined up to isomorphism of categories (see below G.1.2) and no two skeleton objects are isomorphic.

Example 118 Common examples of categories are the categories **Mon** of monoids, **Grp** groups, **Rings** of rings with ring homomorphisms, $\mathbf{LinMod}_R$ R-modules with linear homomorphisms, **LinMod** modules with dilinear homomorphisms, $\mathbf{Mod}_R$ R-modules with affine homomorphisms, **Mod** of modules with diaffine homomorphisms, or **Top** of topological spaces with continuous maps.

Example 119 For every category $\mathbf{C}$ we have the *opposite category* $\mathbf{C}^{opp}$. Its morphisms are the same, but composition works via $f \circ^{opp} g = g \circ f$, i.e., it is defined iff the composition with opposite factors is defined in $\mathbf{C}$. This opposite construction exchanges the domains and codomains of morphisms. Intuitively, an arrow $f : x \to y$ in $\mathbf{C}$ becomes a arrow $f : y \to x$ in $\mathbf{C}^{opp}$.

G.1.2 Functors

Functors are the morphisms between categories:

Definition 164 *If $\mathbf{C}, \mathbf{D}$ are categories, a functor $F : \mathbf{C} \to \mathbf{D}$ is a function which assigns to every morphism c in $\mathbf{C}$ a morphism $F(c)$ in $\mathbf{D}$ such that*

(i) $F(c)$ is an identity if c is so,
(ii) if $c \circ c'$ is defined in $\mathbf{C}$, then $F(c) \circ F(c')$ is defined and $F(c \circ c') = F(c) \circ F(c')$.

In particular, functors carry isomorphisms to isomorphisms. Moreover, the composition $F \circ G : \mathbf{C} \to \mathbf{E}$ of two functors $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{E}$ is a functor. Two categories $\mathbf{C}, \mathbf{D}$ are called *isomorphic* if there exists a functor isomorphism, i.e., two functors $F : \mathbf{C} \tilde{\to} \mathbf{D}, F^{-1} : \mathbf{D} \tilde{\to} \mathbf{C}$ such that $F^{-1} \circ F = Id_{\mathbf{C}}, F \circ F^{-1} = Id_{\mathbf{C}}$. A functor is called *full* iff the $F(Hom_{\mathbf{C}}(x, y)) = Hom_{\mathbf{D}}(F(x), F(y))$ for all object pairs c, d. It is called *faithful* iff $F : Hom_{\mathbf{C}}(x, y) \to Hom_{\mathbf{D}}(F(x), F(y))$ is injective for all pairs c, d. It is called fully faithful iff it is full and faithful, i.e., the map $F : Hom_{\mathbf{C}}(x, y) \to Hom_D(F(x), F(y))$ is a bijection. Functors are also called "covariant" since they are opposed to functors $F : \mathbf{C}^{opp} \to \mathbf{D}$, which are called "contravariant" but then also denoted by $F : \mathbf{C} \to \mathbf{D}$.

One often considers systems of morphisms in a category $\mathbf{C}$, which are defined by a graphical approach: diagrams. Here is the precise definition.

Definition 165 *A* diagram *in a category $\mathbf{C}$ is a functor $\Delta : P(Q) \to \mathbf{C}$, where Q is a quiver. The diagram Δ is said to commute with respect to a relation $\sim$ among Q-paths, iff Δ factorizes through $P(Q)/\sim$. If the relation is* maximal (*it identifies all paths having common domain and codomain*)*, then the diagram is said to be* commutative *without specification of $\sim$.*

By the very definition of a path category, diagrams are given by systems of morphisms in $\mathbf{C}$ which cope with the domain-codomain configuration in the underlying quiver (i.e., diagram scheme).

Example 120 If $\mathbf{C}$ is a category, a subcategory is a sub-collection $\mathbf{C}'$ of $\mathbf{C}$ such that for each morphism f in $\mathbf{C}'$, its domain and codomain are also in $\mathbf{C}'$, and such that for any two f, g in $\mathbf{C}'$ such that $f \circ g$ is defined in $\mathbf{C}$, the composition is also a morphism in $\mathbf{C}'$. A category can be defined by an arbitrary selection of objects (identities) out of $\mathbf{C}$ and the full collections of morphisms having these identities as domains or codomains.

[3] See Example 120 below.

Such a subcategory is called a *full* subcategory of $\mathbf{C}$. A subcategory obviously induces an *embedding* functor $\mathbf{C}' \to \mathbf{C}$ by the identity on the morphisms in $\mathbf{C}'$. For any collection S of morphisms in $\mathbf{C}$, the smallest subcategory of $\mathbf{C}$ containing S is denoted by $\langle S \rangle$ and called the subcategory *generated by* S.

Example 121 If $\mathbf{C}, \mathbf{D}$ are two categories, the product category $\mathbf{C} \times \mathbf{D}$ consists of all ordered pairs (c, d) of morphisms c in $\mathbf{C}$ and d in $\mathbf{D}$. The composition $(c, d) \circ (c', d')$ is possible iff it is possible in each component and then evaluates to $(c, d) \circ (c', d') = (c \circ c', d \circ d')$. One has the canonical projection functors $p_1 : \mathbf{C} \times \mathbf{D} \to \mathbf{C}, p_2 : \mathbf{C} \times \mathbf{D} \to \mathbf{D}$ with $p_1(c, d) = c, p_2(c, d) = d$. The same procedure allows the definition of any finite product of categories.

G.1.3 Natural Transformations

Natural transformations are the morphisms between functors.

Definition 166 *If $F, G : \mathbf{C} \to \mathbf{D}$ are two functors, a natural transformation $t : F \to G$ is a system of morphisms $t(c) : F(c) \to G(c)$ in $\mathbf{D}$, for each object c in $\mathbf{C}$, such that for every morphism $f : x \to y$ in $\mathbf{C}$, we have $G(f) \circ t(x) = t(y) \circ F(f)$. One can also rephrase this property by requiring the following commutative diagram in $\mathbf{D}$:*

$$\begin{array}{ccc} F(x) & \xrightarrow{t(x)} & G(x) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(y) & \xrightarrow{t(y)} & G(y) \end{array} \tag{G.1}$$

Natural transformations can be composed in an evident way, and the composition is associative. For every functor F we have the natural identity Id_F. We therefore have the category $Func(\mathbf{C}, \mathbf{D})$ of functors $F : \mathbf{C} \to \mathbf{D}$ and natural transformations $Nat(F, G)$ between two functors $F, G : \mathbf{C} \to \mathbf{D}$. Properties between such functors are said to be *natural* if they relate to the category $Func(\mathbf{C}, \mathbf{D})$, for example, $F \tilde{\to} G$ is a natural isomorphism iff it is an isomorphism among the natural transformations from F to G.

If two categories $\mathbf{C}, \mathbf{D}$ satisfy the following properties, they are called *equivalent*, equivalence is an equivalence relation which is weaker than isomorphism.

Lemma 99 *For categories $\mathbf{C}, \mathbf{D}$ the following properties are equivalent:*

(*i*) *There are two functors $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$ such that $G \circ F \tilde{\to} Id_{\mathbf{C}}$ and $F \circ G \tilde{\to} Id_{\mathbf{D}}$, where these isomorphisms are natural.*

(*ii*) *There is a functor $F : \mathbf{C} \to \mathbf{D}$ which is fully faithful and* essentially surjective*, i.e., every object (identity) in ${}_0\mathbf{D}$ is isomorphic to an image $F(c)$ of an object of $\mathbf{C}$.*

Example 122 If $\mathbf{C}$ is a category with sets as hom collections $Hom(x, y)$, we have two types of hom functors as follows: For fixed object x, we have the functor $Hom(x, ?) : \mathbf{C} \to \mathbf{Ens} : y \mapsto Hom(x, y)$, which sends a morphism $f : y \to z$ to $Hom(x, f) : Hom(x, y) \to Hom(x, z) : u \mapsto f \circ u$. We further have the contravariant functor $Hom(?, y) : \mathbf{C}^{opp} \to \mathbf{Ens} : x \mapsto Hom(x, y)$, which sends a morphism $f : x \to z$ to $Hom(f, y) : Hom(z, y) \to Hom(x, y) : u \mapsto u \circ f$. The category $Func(\mathbf{C}^{opp}, \mathbf{Ens})$ of contravariant set-valued functors on $\mathbf{C}$ is denoted by $\mathbf{C}^{@}$; its elements are called (*set-valued*) *presheaves over* C. In the theory of denotators, one works with $\mathbf{Mod}^{@}$ and for a module M, we have the notation $Hom_{\mathbf{Mod}}(M, ?) = M@$, whereas the contravariant hom functor is $Hom_{\mathbf{Mod}}(?, M) = @M$.

Example 123 For two categories $\mathbf{C}$ and $\mathbf{D}$ and an object S of $\mathbf{D}$, we have the constant functor $[S] : \mathbf{C} \to \mathbf{D}$ with $[S](X) = S$ and $[S](f) = Id_S$ for all $X \in Ob(\mathbf{C})$ and all $f \in Mor(\mathbf{C})$. In particular, if S is a set, then we write $[S]$ for the constant functor in $\mathbf{Mod}^{@}$ if the contrary is not stressed.

Given a quiver G, if we fix an object c in a category $\mathbf{C}$, we have the constant diagram $\Delta_c = [c]$. It associates every vertex of G with c and every arrow with Id_c. For a diagram Δ in $\mathbf{C}$, a natural transformation

$[c] \to \Delta$ is called a *cone* on Δ, whereas a natural transformation $\Delta \to [c]$ is called a *cocone on* Δ. In a cone, all arrows starting from c must commute with the arrows of the diagram, whereas in a cocone all arrows arriving at c must commute with the arrows of the diagram.

G.2 The Yoneda Lemma

For a given category $\mathbf{C}$ with sets as hom collections $Hom(x, y)$, the *Yoneda embedding* Y is the functor

$$Y : \mathbf{C} \to \mathbf{C}^{@} : x \mapsto Y(x) = Hom(?, x) \tag{G.2}$$

with the natural transformations $Y(f : x \to y) : Y(x) \to Y(y)$ being defined by $u \mapsto f \circ u$ for $u : z \to x \in Y(x)(z) = Hom(z, x)$. For $\mathbf{C} = \mathbf{Mod}$, we also write $Y = ?@$, i.e., $Y(M) = @M$ for a module M.

A functor F in $\mathbf{C}^{@}$ is called *representable* iff there is an object c in ${}_0\mathbf{C}$ such that $F \tilde{\to} Y(c)$. Yoneda's Lemma states that the full subcategory of representable functors in $\mathbf{C}^{@}$ is equivalent to $\mathbf{C}$, and that such an equivalence is given by the Yoneda embedding. More precisely:

Lemma 100 *For every functor F in $\mathbf{C}^{@}$ and object c in ${}_0\mathbf{C}$, the map*

$$\epsilon : Nat(Y(c), F) \to F(c) : h \mapsto h(c)(Id_c) \tag{G.3}$$

is a bijection.

The proof is an easy exercise, but see also [395, 637]. In particular, if $F = Y(d)$, we have a bijection $\epsilon : Nat(Y(c), Y(d) \tilde{\to} Hom(c, d)$. More precisely, this means that the Yoneda functor Y is fully faithful, so we obtain an equivalence of categories as announced. For $\mathbf{C} = \mathbf{Mod}$, we also write $F(A) = A@F$, even if F is not representable. We then have the bijection $Nat(@A, F) \tilde{\to} A@F$. This means that the evaluation of F at "address" A is the same as the calculation of the morphisms from $@A$ to F. This is a justification of the name "address" for the argument A: Evaluating F at A means "observing F under all morphisms" when being "positioned on (the functor $@A$ of) A". And the Yoneda philosophy means that F is known, when it is known while observed from all addresses.

G.2.1 Universal Constructions: Adjoints, Limits, and Colimits

Definition 167 *We suppose that for two categories $\mathbf{C}, \mathbf{D}$, the hom collections are sets. Given two functors $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$, we say that F is* left adjoint to G *or (equivalently) that G* is right adjoint to F*, in signs $F \dashv G$ iff the functors $Hom_D(F(?), ?) : \mathbf{C}^{opp} \times \mathbf{D} \to \mathbf{Ens}$ and $Hom_{\mathbf{C}}(?, G(?)) : \mathbf{C}^{opp} \times \mathbf{D} \to \mathbf{Ens}$ are isomorphic. One also writes this fact in these symbols:*

$$\frac{c \to G(d)}{F(c) \to d}$$

meaning that morphisms in the numerator correspond one-to-one to morphisms in the denominator.

In particular, if we are given an adjoint pair of functors $F \dashv G$, when fixing the variable d in $\mathbf{D}$, the adjointness isomorphism means that the contravariant functor $c \mapsto Hom_{\mathbf{D}}(F(c), d)$ is representable by the object $G(d)$.

Example 124 For $\mathbf{C} = \mathbf{D} = \mathbf{Ens}$, fix a set A. We have the functors $A \times ? : \mathbf{Ens} \to \mathbf{Ens} : X \mapsto A \times X$ and $?^A : \mathbf{Ens} \to \mathbf{Ens} : X \mapsto X^A$, which are an adjoint pair $A \times ? \dashv ?^A$ via the isomorphism that sends $f : A \times X \to B$ to $f \dashv : X \to B^A : f \dashv (x)(a) = f(a, x)$. This adjointness property is crucial in the definition of exponential objects in topoi. The "exponential set" B^A represents the functor $X \mapsto Hom(A \times X, B)$. See Section G.3.2 for this subject.

A *terminal* object 1 in a category $\mathbf{C}$ is one that admits exactly one morphism, denoted by $! : x \to 1$ from each object x of $\mathbf{C}$. An *initial* object 0 is a terminal object in the opposite category. For example, in **Ens**, every singleton, such as $1 = \{0\}$, is a terminal object, while the empty set 0 is initial.

Example 125 A terminal object in $\mathbf{C}^{@}$ is defined by the constant $1_{\mathbf{C}^{@}} = [1]$ (of the set 1). For a presheaf $P \in \mathbf{C}^{@}$, a *global section* γ is a natural transformation $\gamma : 1_{\mathbf{C}^{@}} \to P$. In other words, the global sections Γ correspond to the hom functor $\Gamma(P) = Nat(1_{\mathbf{C}^{@}}, P)$. The global section functor $\Gamma : \mathbf{C}^{@} \to \mathbf{Ens}$ is right adjoint to the constant functor $[\,] : \mathbf{Ens} \to \mathbf{C}^{@}$.

Universal objects in categories, such as limits and colimits, are related to terminal or initial objects as follows. Given a "basis" object b of $\mathbf{C}$, the *comma category* $\mathbf{C}/b$ has all morphisms $f : x \to b$ as objects, and for two objects $f : x \to b, g : y \to b$, we have

$$Hom_{\mathbf{C}/b}(f, g) = \{u | g \circ u = f\},$$

the set of "commutative triangles above b" with the evident composition. The *cocomma category* $\mathbf{C}/^{opp}b$ is the comma category $(\mathbf{C}^{opp}/b)^{opp}$, in other words, for two objects $f : b \to x, g : b \to y$, we have

$$Hom_{\mathbf{C}/^{opp}b}(f, g) = \{u | u \circ f = g\}.$$

Given a quiver Q and a category $\mathbf{C}$, we have in the category $Func(P(Q), \mathbf{C})$ of diagrams in $\mathbf{C}$, and given such a diagram Δ, the comma category $Func(P(Q), \mathbf{C})/\Delta$. In this category, take the full subcategory $cones(\Delta)$ of cones $[c] \to \Delta$. Then a *limit of* Δ is a terminal object $lim(\Delta)$ in $cones(\Delta)$. Since terminal objects are evidently unique up to isomorphisms, a limit is also unique up to isomorphism. A *colimit* $colim(\Delta)$ *of a diagram* Δ is an initial object in the subcategory $cocones(\Delta)$ of cocones on Δ in the cocomma category $Func(P(Q), \mathbf{C})/^{opp}\Delta$.

If the diagram is a pair $f : a \to c, g : b \to c$, the limit is called the *fiber product* or *pullback* of f, g, or (more sloppily) of a and b if f, g are clear; it is denoted by $a \times_c b$. If the diagram is a pair $f : c \to a, g : c \to b$, the colimit is called the *fiber sum* or *pushout* of f, g, or (more sloppily) of a and b if f, g are clear; it is denoted by $a \sqcup_c b$. The limit of two isolated objects a, b (discrete diagram with two points) is called the (*cartesian*) *product of* a, b and denoted by $a \times b$. The colimit of two isolated objects a, b is called the (*disjoint*) *sum of* a, b and denoted by $a \sqcup b$.

Theorem 68 *The category of sets* **Ens** *has arbitrary limits and colimits. For a category* $\mathbf{C}$ *with sets as hom collections, the category* $\mathbf{C}^{@}$ *of presheaves over* $\mathbf{C}$ *has arbitrary limits and colimits.*

Proof. If Δ is a diagram of set morphisms $f_{i,j,k} : X_i \to X_j$, the limit is the subset in $\prod_i X_i$ consisting of all families (x_i) such that for any pair $(x_i, x_j) \in X_i \times X_j$ and any $f_{i,j,k}$, we have $f_{i,j,k}(x_i) = x_j$. The projections $lim(\Delta) \to X_i$ are the restrictions of the canonical projections from the product to X_i. The colimit is the set $colim(\Delta)$ of equivalence classes $\coprod_i X_i / \sim$ defined by the equivalence relation generated by the relation $x_i \sim x_j$ iff there is $f_{i,j,k}(x_i) = x_j$. The morphisms $X_i \to colim(\Delta)$ are the injections $X_i \to \coprod_i X_i$, followed by the quotient map $\coprod_i X_i \to colim(\Delta)$. The universal properties are immediate and left as an exercise.

For a diagram Δ of presheaves F_i, we take for each argument c in C the set-theoretic limit or colimit, respectively, of the set diagram $\Delta(c)$ of the sets $F_i(c)$ and the corresponding maps to define the limit or colimit of Δ, respectively, QED.

The following proposition makes sure that the category of presheaves over $\mathbf{C}$ is not too large with respect to its Yoneda embedding of $\mathbf{C}$:

Proposition 109 *Every presheaf* F *in* $\mathbf{C}^{@}$ *is a colimit of representable presheaves.*

See [639, pp.41/42] for a proof. The idea of this proof uses the so-called *category of elements* $\int_{\mathbf{C}} F$ of a functor F. Its objects are all pairs (C, p) where C is an object of $\mathbf{C}$, and $p \in C@F$. The morphisms $(C, p) \to (C', p')$ are the morphisms $u : C \to C'$ in $\mathbf{C}$ such that $p'.u = p$.

Definition 168 *A category is called* ⟨finitely⟩ (co)complete *iff it has* (*co*)*limits for all* ⟨*finite*⟩ *diagrams* ⟨*diagrams with finitely many objects and arrows*⟩.

Proposition 109 turns out to make the Yoneda embedding into a universal device for making a category **C** cocomplete:

Proposition 110 *For each functor* $f : \mathbf{C} \to \mathbf{E}$ *to a cocomplete category* **E**, *there exists an essentially unique colimit preserving functor* $L : \mathbf{C}^{@} \to \mathbf{E}$ *such that* $f = L \circ Y$.

See [639, p.43] for a proof.

G.2.2 Limit and Colimit Characterizations

Proposition 111 *For any category* **C**, *the following statements are equivalent:*

(*i*) **C** *is finitely complete.*
(*ii*) **C** *has finite products and equalizers*[4].
(*iii*) **C** *has a terminal object and fiber products.*

For a proof, see [953, I, 7.8.8].

Proposition 112 *For any category* **C**, *the following statements are equivalent:*

(*i*) **C** *is finitely cocomplete.*
(*ii*) **C** *has finite sums and coequalizers*[5].
(*iii*) **C** *has an initial object and fiber sums.*

This is just the dual statement of Proposition 111.

Proposition 113 *Let* **C** *be a finitely complete category. A morphism* $f : A \to B$ *in* **C** *is mono iff the canonical projections* p_1 *and* p_2 *in the pullback*

$$\begin{array}{ccc} X & \xrightarrow{p_1} & A \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array} \tag{G.4}$$

coincide and are isomorphisms.

Proof. Clearly, if f is mono, then $X = A$ and $p_1 = p_2 = 1_A$ define a fiber product. Conversely, if $p_1 = p_2$ is an isomorphism, then any couple $u, v : Z \to A$ with $f \cdot u = f \cdot v$ creates factorizations $u = p_1 \cdot t$ and $v = p_2 \cdot t$ through $t : Z \to X$ which therefore also coincide. QED.

Therefore we have the dual result:

Corollary 36 *Let* **C** *be a finitely cocomplete category. A morphism* $f : A \to B$ *in* **C** *is epi iff the canonical morphisms* i_1 *and* i_2 *in the pushout*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i_1} \\ B & \xrightarrow{i_2} & X \end{array} \tag{G.5}$$

coincide and are isomorphisms.

[4] An equalizer is a limit of a pair $f, g : x \rightrightarrows y$ of arrows.
[5] A coequalizer is a colimit of a pair $f, g : x \rightrightarrows y$ of arrows.

Proposition 114 *For any category* $\mathbf{C}$*, let* $f : H \to G$ *be a morphism in* $\mathbf{C}^{@}$*. Then:*

(*i*) *The morphism* f *is mono iff* $A@f : A@H \to A@G$ *is injective for all objects* A *of* $\mathbf{C}$*.*
(*ii*) *The morphism* f *is epi iff* $A@f : A@H \to A@G$ *is surjective for all objects* A *of* $\mathbf{C}$*.*
(*iii*) *The morphism* f *is iso iff it is mono and epi iff* $A@f : A@H \to A@G$ *is bijective for all objects* A *of* $\mathbf{C}$*.*

Proof. Observe that $\mathbf{C}^{@}$ is finitely complete and cocomplete and that limits and colimits are calculated pointwise. Let us first look at point (iii). Clearly, f is iso iff its evaluations $A@f : A@H \to A@G$ are all bijective. Further, we know from Proposition 113 that f is mono iff the fiber product projections p_1 and p_2 coincide and are iso. But with (iii) this is true iff this is true for all evaluations at objects A of $\mathbf{C}$, i.e., iff this is true set-theoretically, and this means having an injection for every object A of $\mathbf{C}$, and (i) is done; the dual argument shows (ii). Finally, iso always implies mono and epi; conversely, mono and epi means being in- and surjective, i.e., bijective at every object A of $\mathbf{C}$, whence f is iso. QED.

G.2.2.1 Special Results for $\mathbf{Mod}^{@}$

Lemma 101 *Let* H *be a functor in* $\mathbf{Mod}^{@}$*,* M, N *be two addresses, and* $f : N \to M$ *a morphism of addresses. Then the map*

$$M \mapsto M@Fin(H) := \{F \subset M@H, card(F) < \infty\}, \tag{G.6}$$

together with the maps

$$f@Fin(H) : M@Fin(H) \to N@Fin(H) : X \mapsto f@H(X), \tag{G.7}$$

defines a functor $Fin(H)$ *in* $\mathbf{Mod}^{@}$*.*

Lemma 102 *For any functor* H *in* $\mathbf{Mod}^{@}$ *and address* M*, the maps*

$$sing_H(M) : M@H \to M@Fin(H) : x \mapsto \{x\} \tag{G.8}$$

defines a monomorphism $sing_H : H \rightarrowtail Fin(H)$ *of functors.*

Lemma 103 *The map*

$$Fin : \mathbf{Mod}^{@} \to \mathbf{Mod}^{@} : H \mapsto Fin(H) \tag{G.9}$$

defines an endofunctor on $\mathbf{Mod}^{@}$*, and the monomorphism sing defines a natural transformation*

$$sing : Id_{\mathbf{Mod}^{@}} \rightarrowtail Fin. \tag{G.10}$$

Lemma 104 *Let* $\mathbf{D} = H_0 \underset{f_0}{\to} H_1 \underset{f_1}{\to} H_2 ...$ *be a natural sequence diagram in* $\mathbf{Mod}^{@}$*. Then we have*

$$colim(Fin(\mathbf{D})) \tilde{\to} Fin(colim(\mathbf{D}). \tag{G.11}$$

This yields an important proposition for the construction of circular forms and denotators.

Proposition 115 *Let* H *be a functor in* $\mathbf{Mod}^{@}$*. Then there are functors* X *and* Y *in* $\mathbf{Mod}^{@}$ *such that*

$$X \tilde{\to} Fin(H \times X) \text{ and} \tag{G.12}$$
$$Y \tilde{\to} H \times Fin(Y). \tag{G.13}$$

Proof. For the first isomorphism, let $(X_n)_{0\leqslant n}$ be the following sequence of functors in $\mathbf{Mod}^{@}$. We recursively define

$$X_n = \begin{cases} \varnothing & \text{for } n = 0, \\ Fin(H \times X_{n-1}) & \text{for } n > 0. \end{cases} \tag{G.14}$$

Then we have a diagram of subfunctors

$$f_n : X_n \hookrightarrow X_{n+1} \tag{G.15}$$

for all $0 \leqslant n$. In fact, clearly $X_0 \hookrightarrow X_1$. Now, let $0 < n$ and take an address M. We have $M@X_n = M@Fin(H\times X_{n-1}) = Fin(M@H\times M@X_{n-1})$. Since by induction $X_{n-1} \hookrightarrow X_n$, we have $M@X_{n-1} \subset M@X_n$ and hence $M@X_n \subset M@X_{n+1}$. Now, we know from [953] that the product commutes with the colimit over a sequence diagram. Taking the diagram $\mathbf{D} = X_0 \underset{f_0}{\to} X_1 \underset{f_1}{\to} X_2...$ and setting $X = colim(\mathbf{D})$, Lemma 104 yields

$$\begin{aligned} & Fin(H \times X) = \\ & Fin(H \times colim(\mathbf{D})) \xrightarrow{\sim} \\ & Fin(colim(H \times \mathbf{D})) \xrightarrow{\sim} \\ & colim(Fin(H \times \mathbf{D})) \xrightarrow{\sim} \\ & colim(\mathbf{D}) = \\ & X \end{aligned} \tag{G.16, G.17}$$

and we are done for the first isomorphism. For the second, take

$$Y_n = \begin{cases} \varnothing & \text{for } n = 0, \\ H \times Fin(Y_{n-1}) & \text{for } n > 0. \end{cases} \tag{G.18}$$

We again have a diagram $\mathbf{E} = Y_0 \underset{g_0}{\to} Y_1 \underset{g_1}{\to} Y_2...$ of subfunctors

$$g_n : Y_n \hookrightarrow Y_{n+1} \tag{G.19}$$

for all $0 \leqslant n$. Setting $Y = colim(\mathbf{E})$, our second isomorphism results:

$$\begin{aligned} & H \times Fin(Y) = \\ & H \times Fin(colim(\mathbf{E})) \xrightarrow{\sim} \\ & H \times colim(Fin(\mathbf{E})) \xrightarrow{\sim} \\ & colim(H \times Fin(\mathbf{E})) \xrightarrow{\sim} \\ & colim(\mathbf{E}) = \\ & Y \end{aligned} \tag{G.20, G.21}$$

and we are done.

G.3 Topoi

Topoi are special categories which imitate the crucial constructions of set theory, such as cartesian products, disjoint unions, power sets, and characteristic maps. In our context, topoi play two roles: (1) the role of basic mathematical realities which are instantiated to get off ground in denotator theory, i.e., to build compound concept spaces and their points; (2) the more technical role of topoi of sheaves associated with presheaves for Grothendieck topologies.

G.3.1 Subobject Classifiers

Definition 169 *Given a category* $\mathbf{C}$ *which is finitely complete, with the terminal object* 1*, a monomorphism* $true : 1 \rightarrowtail \Omega$ *in* $\mathbf{C}$ *is called a* subobject classifier *iff given any monomorphism* $\sigma : S \rightarrowtail X$ *in* $\mathbf{C}$*, there is a unique morphism* $\chi_\sigma : X \to \Omega$ *such that the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & X \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \chi_\sigma} \\ 1 & \xrightarrow{true} & \Omega \end{array} \tag{G.22}$$

is a pullback.

Subobject classifiers are unique up to isomorphism. If a subobject classifier exists, the morphism χ must be the same if we replace σ by $\sigma \circ q$ for any isomorphism $q : S' \xrightarrow{\sim} S$ since an isomorphic object to a pullback is also a pullback. A *subobject of* X is an equivalence class of monomorphisms $\sigma : S \rightarrowtail X$ under the relation $\sigma \sim \sigma'$ iff there is an isomorphism q such that $\sigma' = \sigma \circ q$. Suppose that the collection of subobjects of X is a set $Sub_{\mathbf{C}}(X)$ for each object X in $\mathbf{C}$. Then this is a presheaf in $\mathbf{C}^{@}$ by this map: take a morphism $f : Y \to X$. Then we define

$$Sub_{\mathbf{C}}(f) : Sub_{\mathbf{C}}(X) \to Sub_{\mathbf{C}}(Y) : \sigma \mapsto \sigma_f$$

where $\sigma_f : S \times_X Y \to Y$ is the canonical projection of the pullback under f, σ. It is straightforward that this is a monomorphism. Then we have:

Proposition 116 *A category* $\mathbf{C}$ *which is finitely complete and such that the subobject presheaf* $Sub_{\mathbf{C}}$ *is defined, has a subobject classifier iff the* $Sub_{\mathbf{C}}$ *is representable,* $Sub_{\mathbf{C}}(X) \xrightarrow{\sim} Hom(X, \Omega)$ *for all* X*. If so, the subobject classifier can be set to the inverse image* $\epsilon \rightarrowtail \Omega$ *of* Id_Ω *in* $Sub_{\mathbf{C}}(\Omega)$*.*

See [639, p.33] for a proof.

Example 126 In the category **Ens**, the ordinal number inclusion $true : 1 \rightarrowtail 2 = \{0, 1\} : 0 \mapsto 0$ is a subobject classifier. Since the subobjects of a set X in **Ens** identify to the subsets $S \subseteq X$, we have the classical result that subsets S of X are characterized by their characteristic maps $\chi_S : X \to 2$, a fact that is also traced in the notation 2^X for the set of subsets, the powerset of X.

Example 127 The equivalence classes of monomorphisms of presheaves $S \rightarrowtail X$ in $\mathbf{C}^{@}$ are defined by their images $Im(S) \subseteq X$ (take everything pointwise). So $sub_{\mathbf{C}^{@}}(X) \xrightarrow{\sim} \{S \subseteq X\}$, the set of subfunctors of X (supposing that it exists as a set). By the Yoneda Lemma, if a subobject classifier in $\mathbf{C}^{@}$ exists, we must have[6] $Sub_{\mathbf{C}^{@}}(@Y) \xrightarrow{\sim} Hom(@Y, \Omega) \xrightarrow{\sim} \Omega(Y)$. So the functor $Y \mapsto Sub_{\mathbf{C}^{@}}(@Y)$ is a canonical candidate for Ω, and it in fact does the job, see [639, pp.37/38]. The final presheaf being the constant presheaf $1_{\mathbf{C}^{@}} : X \mapsto 1$, we get the *true* morphism (natural transformation) $true(0) = @Y$. A subfunctor of $@Y$ is called a *sieve in* Y, so a candidate for the *subobject classifier is the functor of sieves* (verify that it is a functor!).

Exercise 90 *The categories* **Ab** *of abelian groups or* $\mathbf{Mod}_R$ *of R-modules have no subobject classifiers.*

In denotator theory, sieves and more general subfunctors replace local compositions (which are essentially subsets of ambient modules) in the functorial setup. This is also necessitated since module categories are no topoi (since they have no subobject classifiers, see definition 170 in Appendix G.3.3), so the passage to the presheaves over modules, i.e., the category $\mathbf{Mod}^{@}$ is mandatory in order to recover the subobject classifier structure.

[6] Writing the shorter $@Y$ instead of $Hom(?, Y)$.

G.3.2 Exponentiation

Recall Example 124 in Appendix G.2.1 of exponential sets. More generally, a category **C** is called *cartesian closed* iff it has finite products[7] and each element A is *exponentiable*, which means that the functor $A\times ?$ has a right adjoint $?^A$, i.e., we have an adjoint pair of functors $A\times ? \dashv ?^A$.

Example 128 The category of sets **Ens** is cartesian closed. And any product of cartesian closed categories is cartesian closed.

Example 129 A category of presheaves $\mathbf{C}^{@}$ is cartesian closed by the following discussion. Again, we use the Yoneda Lemma to find a canonical candidate of the exponentiation X^Y of two presheaves X, Y. If the exponential X^Y exists, we must have $U@X^Y \tilde{\to} Nat(@U, X^Y) \tilde{\to} Nat(@U \times Y, X)$. So one canonical definition must be

$$U@X^Y = Nat(@U \times Y, X) \tag{G.23}$$

for any object U in **C**, which is evidently a presheaf. The proof that this formula does the job is found in [639, p.47].

In every cartesian closed category **C** one has these standard formulas

$$1^X \tilde{\to} 1, X^1 \tilde{\to} X, (Y \times Z)^X \tilde{\to} Y^X \times Z^X, X^{Y\times Z} \tilde{\to} (X^Y)^Z,$$

which follow from the universal adjointness property of exponentiation.

G.3.3 Definition of Topoi

There are several equivalent definitions of a topos, which we first summarize in the following proposition:

Proposition 117 *For a category* **C***, the following group properties are equivalent:*

1. *a)* **C** *is cartesian closed,*
   b) **C** *has a subobject classifier* $1 \rightarrowtail \Omega$.
2. *a)* **C** *is cartesian closed,*
   b) **C** *is finitely cocomplete,*
   c) **C** *has a subobject classifier* $1 \rightarrowtail \Omega$.
3. *a)* **C** *has a terminal object and pullbacks,*
   b) **C** *has exponentials,*
   c) **C** *has a subobject classifier* $1 \rightarrowtail \Omega$.
4. *a)* **C** *has a terminal object and pullbacks,*
   b) **C** *has an initial object and pushouts,*
   c) **C** *has exponentials,*
   d) **C** *has a subobject classifier* $1 \rightarrowtail \Omega$.
5. *a)* **C** *is finitely complete,*
   b) **C** *has power objects*[8].

Definition 170 *A category* **C** *which has the equivalent groups of properties in Proposition 117 is called a* (elementary) topos.

Here are some general properties and examples of topoi:

Proposition 118 *For a topos* **C** *a comma category* $\mathbf{C}/b$ *is also a topos.*

[7] Equivalently: binary products and a terminal object.
[8] See [376, p.106] for this group of properties.

Proposition 119 *For a category* $\mathbf{C}$ *the presheaf category* $\mathbf{C}^{@}$ *is a topos.*

This is immediate from the previous discussion of the presheaf category.

Proposition 120 *Let* $\mathbf{C}$ *be a topos. Then we have these properties:*

(*i*) *Every morphism* f *has an image, i.e., factors as* $f = i \circ e$ *with* i *mono and* e *epi. For any two such factorizations* $f = i \circ e, f = i' \circ e'$*, there is an isomorphism* t *such that* $e' = t \circ e, i' = i \circ t$*.*
(*ii*) *A morphism is iso iff it is mono and epi.*
(*iii*) *The pullback of an epi is an epi.*
(*iv*) *Every arrow* $X \to 0$ *is iso.*
(*v*) *Every arrow* $0 \to X$ *is mono.*

Definition 171 Logical *morphisms between topoi are functors which preserve* (*up to isomorphism*) *finite limits, exponentials, and subobject classifiers.*

For example, the canonical base change functor $\mathbf{C}/b \to \mathbf{C}/c$ of comma topoi for a base change morphism $c \to b$ is logical, see [639, p.193].

G.4 Grothendieck Topologies

Grothendieck topologies and associated topoi of sheaves are a classical example for the geometric aspects of topoi. Here is the context.

Given a finitely complete category (a small one for those who like universes) $\mathbf{C}$ with the subobject classifier functor of sieves $X@\Omega = Sub_{\mathbf{C}^{@}}(@X)$ (see Example 127). Recall that given a morphism $f : Y \to X$ the functor maps a sieve $S \subseteq X$ to the pullback sieve $f^*(S) = S \times_X Y$.

Definition 172 *A* Grothendieck topology *on a category* $\mathbf{C}$ *is a function* J *which for each* X *is a subset* $X@J \subseteq X@\Omega$ *of sieves in* X *with these properties:*

(*i*) $@X \in X@J$,
(*ii*) *(Stability) If* $S \in X@J$*, then for* $f : Y \to X$*,* $f^*S \in Y@J$*,*
(*iii*) *(Transitivity) If* $S \in X@J$ *and* $R \in X@\Omega$ *with* $f^*R \in Y@J$ *for all* $f : Y \to X$ *in* S*, then* $R \in X@J$*.*

A site *is a pair* $(\mathbf{C}, J)$ *of a Grothendieck topology* J *on a category* $\mathbf{C}$*. A sieve in* $X@J$ *is called a* covering sieve*, one also says that "it covers* X*".*

The first two requirements mean that J is a subfunctor of Ω through which the *true* arrow factorizes. Very often, Grothendieck topologies are not given directly, but via a so-called basis:

Definition 173 *For a finitely complete category* $\mathbf{C}$*, a* basis (for a Grothendieck topology) *is a function* K *which assigns to each object* X *a collection* $K(X)$ *of families of morphisms with codomain* X *such that:*

(*i*) *For every isomorphism* $f : X' \xrightarrow{\sim} X$*, the singleton* $\{f\}$ *is in* $K(X)$*;*
(*ii*) *(Stability) If* $(f_i : X_i \to X) \in K(X)$*, and* $h : Y \to X$*, then* $(h^* f_i : X_i \times_X Y \to Y) \in K(Y)$*;*
(*iii*) *(Transitivity) If* $(f_i : X_i \to X) \in K(X)$ *and, for each index* i*,* $(f_{ij} : X_{ij} \to X_i) \in K(X_i)$*, then* $(f_i \circ f_{ij} : X_{ij} \to X) \in K(X)$*.*

A pair $(\mathbf{C}, K)$ *is again called a* site *(see below for a justification!); whereas the families in the sets* $K(X)$ *are called* covering families*.*

Here is the relation to Grothendieck topologies: Given a basis K as above, one defines

$$J_K(X) = \{S| \text{ there is } R \in K(X) \text{ with } R \subseteq S\}, \tag{G.24}$$

where $R \subseteq S$ means that R is in the union of the evaluations $\bigcup_{\mathbf{C}} Z@S$ of S.

And the converse: Given a family R of morphisms with codomain X, we denote by (R) the sieve generated by R, i.e., the smallest sieve in X containing all arrows of R. Then a Grothendieck topology J can be defined by the following basis K which is this set at X:

$$K(X) = \{R \subseteq @X|(R) \in X@J\}. \tag{G.25}$$

G.4.1 Sheaves

Definition 174 *Given a site $(\mathbf{C}, J)$, a presheaf P in $\mathbf{C}$ is a sheaf for J iff for every covering sieve $S \subseteq @X$, the inclusion induces a bijection $Nat(@X, P) \tilde{\to} Nat(S, P)$.*

This condition can be rephrased for a basis K of J in a more effective and classical way. To this end recall that a (co)equalizer of a pair $f, g : x \rightrightarrows y$ of parallel arrows is the (co)limit of this diagram.

Proposition 121 *A presheaf P on $\mathbf{C}$ is a sheaf for the topology J, iff for any covering family $(f_i : X_i \to X) \in K(X)$, the canonical diagram*

$$P(X) \to \prod_i P(X_i) \rightrightarrows \prod_{i,j} P(X_i \times_X X_j) \tag{G.26}$$

is an equalizer.

Here the two arrows to the right stem from the two projections from $X_i \times_X X_j$ to X_i and to X_j, whereas the arrow to the left stems from the covering morphisms f_i.

Definition 175 *A* Grothendieck topos *is a category which is equivalent to the full subcategory $Sh(\mathbf{C}, J)$ of sheaves in $\mathbf{C}^@$*

The justification of this terminology lies in the following theorem:

Theorem 69 *A Grothendieck topos is an elementary topos.*

The proof evidently splits in the verification of finite completeness, existence of exponentials and of a subobject classifier. For details, see [639, pp.128-144]. Finite completeness is easy since:

Lemma 105 *Limits of sheaves are sheaves.*

Lemma 106 *If P is a presheaf and F is a J-sheaf in $\mathbf{C}^@$, then the presheaf exponential F^P is a sheaf and therefore an exponential in $Sh(\mathbf{C}, J)$.*

Definition 176 *A sieve $L \subseteq @X$ is* closed *with respect to $(\mathbf{C}, J)$ iff $f : Y \to X$ with $f^*(L) \in Y@J$ implies $f \in Y@L$.*

Proposition 122 *The function $X \mapsto X@\Omega_{Sh} = \{$closed sieves in $X\} \subset X@\Omega$ contains $@X$ defines a subpresheaf of the subobject classifier of $\mathbf{C}^@$ and is a sheaf. Together with the morphism true $: 1_{\mathbf{C}^@} \to \Omega_{Sh}$ it defines a subobject classifier of $Sh(\mathbf{C}, J)$.*

The topos $Sh(\mathbf{C}, J)$ is a subtopos of $\mathbf{C}^{@}$ with the natural inclusion $i : Sh(\mathbf{C}, J) \to \mathbf{C}^{@}$. This natural transformation has a left adjoint of sheafification which we shall discuss now. If P is a presheaf over $\mathbf{C}$, the sheafification operator $P \mapsto P^+$ evaluates as follows. For an object $X \in \mathbf{C}$ and a sieve $S \in X@J$, consider the limit $Match_P(S) = \prod_{f:Y\to X \in S} Y@P$. Consider the diagram $(Match_P(S))_{S\in X@J}$ with canonical restriction maps $Match_P(S) \to Match_P(T)$ for $T \subseteq S$, and define $X@P^+ = lim_{S\in X@J} Match_P(S)$. For a morphism $g : X_1 \to X_2$, we have a map $P^+(g) : X_2@P^+ \to X_1@P^+$. It takes a "matching family" $(x_f)_{f\in S}$ to the matching family $(x_{g.h})_{h\in g^\star S}$. This evidently defines a presheaf, and we have a canonical morphism $\eta : P \to P^+$. Then:

Theorem 70 *With the above notation, we have*

(*i*) *The presheaf P^+ is separated*[9]*.*
(*ii*) *The presheaf P is separated iff η is mono.*
(*iii*) *The presheaf P is a sheaf iff η is iso.*

For a proof, see [639, III.5]. In particular, the double application $\mathbf{a}P = ((P)^+)^+$ yields a sheaf and a natural presheaf morphism $P \to \mathbf{a}P$. We have

Theorem 71 *The map $P \mapsto \mathbf{a}P$ defines a left adjoint of the inclusion i; $\mathbf{a} \dashv i$. The composition $\mathbf{a} \circ i$ is isomorphic to the identity on the sheaf category $Sh(\mathbf{C}, J)$.*

For a proof, see [639, III.5, Theorem 1] and [639, III.5, Corollary 6].

Corollary 37 *If $f : F \to G$ is a morphism of sheaves, f is mono iff it is an injection for each argument.*

The proof follows from the fact that this is true for presheaves, and that by the adjunction Theorem 71, i preserves and reflects[10] monomorphisms, QED.

G.5 Formal Logic

Formal logic does not replace absolute logic which is built upon the non-formalizable theorem of identity (A is identical to A), of contradiction (A and non-A exclude each other), and of the excluded third (there is no third choice except A or non-A). It does however model the way a specific domain of knowledge can handle its formal truth mechanisms.

G.5.1 Propositional Calculus

Sentences in propositional calculus are defined from a set $\Phi = \{\pi_0, \pi_1, \ldots\}$ of symbols, called *propositional variables*; a set $\Xi = \{!, \&, |, ->\}$ of logical connective symbols[11] (! for negation, & for conjunction, | for disjunction, and $->$ for implication); a set $\Delta = \{(,)\}$ of two brackets; mutually disjoint from each other. Let $EX = FM(\Phi \cup \Xi \cup \Delta)$ be the free monoid of word *expressions* above these symbols. Let $S(EX)$, the set of *sentences*, be the smallest subset of EX with these properties:

Property 4 *Given the symbol sets Φ, Ξ, Δ, we require:*

(*i*) $\Phi \subset S(EX)$*;*
(*ii*) *if $\alpha \in S(EX)$, then $(!\alpha) \in S(EX)$;*
(*iii*) *if $\alpha, \beta \in S(EX)$, then $(\alpha\&\beta) \in S(EX)$;*

[9] The left arrow in equation (G.26) for P^+ is injective.
[10] A functor f reflects a property of morphisms if the property for the image morphism $f(x)$ implies the property for x.
[11] Our symbols are near to programming symbols, where the formalism is really needed.

(*iv*) *if* $\alpha, \beta \in S(EX)$, *then* $(\alpha|\beta) \in S(EX)$;
(*v*) *if* $\alpha, \beta \in S(EX)$, *then* $(\alpha -> \beta) \in S(EX)$.

Clearly, in $S(EX)$, the building blocks of a sentence are uniquely determined, so it makes sense to define set-valued functions $\epsilon : S(EX) \to A$ on such sentences by recursion of the building blocks. Suppose that A is a lattice, i.e., a partially ordered set $(A, \leqslant)$ with a *join* operation $\vee : A \times A \to A$, a *meet* operation $\wedge : A \times A \to A$, minimum (False) $\bot$, a maximum (True) $\top$, further a unary *negation* operation $\neg : A \to A$, and a binary *implication* operation $\Rightarrow$: $A \times A \to A$. Call such an A a *logical algebra*. Then any set function $\epsilon_0 : \Phi \to A$ extends in a unique way to the *evaluation* $\epsilon = \epsilon(\epsilon_0) : S(EX) \to A$ by these rules:

Property 5 *For all sentences* α, β, *we set*

(*i*) $\epsilon(!\alpha) = \neg\epsilon(\alpha)$;
(*ii*) $\epsilon((\alpha\&\beta)) = \epsilon(\alpha) \wedge \epsilon(\beta)$;
(*iii*) $\epsilon((\alpha|\beta)) = \epsilon(\alpha) \vee \epsilon(\beta)$;
(*iv*) $\epsilon((\alpha -> \beta)) = \epsilon(\alpha) \Rightarrow \epsilon(\beta)$.

Propositional calculus deals with the evaluation map on special logical algebras. A sentence α *is called* A*-valid*, in symbols: $A \models \alpha$, iff $\epsilon(\alpha) = \top$ for all evaluations $\epsilon_0 : \Phi \to A$ on the propositional variables. It is called *classically valid* or a *tautology* iff it is 2-valid for the well-known Boolean algebra $2 = \{0, 1\}$ of classical truth values, where we set $\top = 1, \bot = 0$. The symbol for classical validity is $\models \alpha$. Here are typical classes of logical algebras:

Boolean Algebras. A Boolean algebra is a distributive logical algebra such that $x \vee \neg x = \top$ and $x \wedge \neg x = \bot$. Distributivity means that $x \wedge (y \vee z) = x \wedge y \vee x \wedge z$ and $x \vee (y \wedge z) = x \wedge y \vee x \wedge z$. Further, implication is defined by $x \Rightarrow y = \neg x \vee y$. In a Boolean algebra (BA), one has these properties: $\neg\neg x = x$, $x \wedge y = \bot$ iff $y \leqslant \neg x$, $x \leqslant y$ iff $\neg y \leqslant \neg x$, $\neg(x \wedge y) = \neg x \vee \neg y$, $\neg(x \vee y) = \neg x \wedge \neg y$.

Heyting Algebras. A Heyting algebra A is a partially ordered set which, as a category whose morphisms $x \to y$ are the pairs $x \leqslant y$, has all finite products and coproducts, and which has exponentials, so it is cartesian closed. In other words, a Heyting algebra is a lattice with minimum $\bot$ and maximum $\top$ which has exponentials x^y. One writes the product as meet ($\wedge$) and the coproduct as join ($\vee$). The exponential x^y is written as $y \Rightarrow x$, and the adjunction property of exponentiation reads

$$z \leqslant y \Rightarrow x \text{ iff } z \wedge y \leqslant x. \tag{G.27}$$

For a Heyting algebra, we define a negation $\neg x = x \Rightarrow \bot$, which is equivalent to $y \leqslant \neg x$ iff $y \wedge x = \bot$.

Proposition 123 *For a Heyting algebra, we have these identities:* $x \leqslant \neg\neg x$, $x \leqslant y$ *implies* $\neg y \leqslant \neg x$, $\neg x = \neg\neg\neg x$, $\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y$, $(x \Rightarrow x) = \top$, $x \wedge (x \Rightarrow y) = x \wedge y$, $y \wedge (x \Rightarrow y) = y$, $x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z)$.

Proposition 124 *A Heyting algebra is distributive, and it is Boolean iff* $x = \neg\neg x$ *for all* x, *or iff* $x \vee \neg x = \top$ *for all* x.

Proposition 125 *For a presheaf category* $\mathbf{C}^{@}$, *the partially ordered set* $Sub_{\mathbf{C}^{@}}(P)$ *of an object* P *is a Heyting algebra. The connectives are defined as follows. If* S, T *are two subfunctors of* P, *then:*
(*i*) $X@(S \vee T) = X@S \cup X@T$;
(*ii*) $X@(S \wedge T) = X@S \cap X@T$;
(*iii*) $X@(S \Rightarrow T) = \{x \in X@P |$ *for every morphism* $f : Y \to X$, *if* $x \cdot f \in Y@S$, *then* $x \cdot f \in Y@T\}$;
(*iv*) $X@(\neg S) = \{x \in X@P |$ *for every morphism* $f : Y \to X, x \cdot f \notin Y@S\}$;

More generally (see [639, IV.8] for details):

Theorem 72 *For every topos* $\mathbf{C}$, *the partially ordered set* $Sub(X) \tilde{\to} Hom(X, \Omega)$ *of subobjects of* X *is a Heyting algebra.*

To the left, this structure stems from the canonical Heyting algebra structure on the subobjects of X. To the right, this structure is induced by the following operations on Ω:

1. Negation $\neg : \Omega \to \Omega$ is the characteristic map of the false arrow $false : 1 \rightarrowtail \Omega$, which is the characteristic map of the zero arrow $0 \rightarrowtail 1$.
2. Disjunction $\wedge : \Omega \times \Omega \to \Omega$ is the character of the diagonal morphism $\Delta(true, true) : 1 \to \Omega$.
3. Conjunction $\vee : \Omega \times \Omega \to \Omega$ is the character of the image of the universal morphism $a \sqcup b : \Omega \sqcup \Omega \to \Omega \times \Omega$, where $a = \Delta(Id_\Omega, true) : \Omega \to \Omega \times \Omega$ and $b = \Delta(true, Id_\Omega) : \Omega \to \Omega \times \Omega$.
4. Implication $\Rightarrow$: $\Omega \times \Omega \to \Omega$ is the character of the equalizer of $p_1, \wedge : \Omega \times \Omega \rightrightarrows \Omega$.

If a sentence α is valid for all Boolean algebras, we write $BA \models \alpha$. If it is valid for all Heyting algebras, we write $HA \models \alpha$. If it is valid in the Heyting algebra $Hom(1, \Omega)$ of a topos $\mathbf{C}$, we write $\mathbf{C} \models \alpha$.

Validity is also described by a recursive construction process of valid sentences. One gives a set AX of sentences, called *axioms*, and defines *theorems* as those sentences s which are at the end of *proof chains*, i.e., finite sequences of sentences $(s_0, s_1, \ldots s_n, s)$ such that each member of this sequence is either an axiom or can be inferred from earlier members by a set $RULES$ of rules.

The classical setup is this. AX consists of 12 types of sentences:

Axiom 7 *Axioms of classical logic (CL):*

(i) α –> $(\alpha\&\alpha)$
(ii) $(\alpha\&\beta)$ –> $(\beta\&\alpha)$
(iii) $(\alpha$ –> $\beta)$ –> $((\alpha\&\gamma)$ –> $(\beta\&\gamma))$
(iv) $((\alpha$ –> $\beta)\&(\beta$ –> $\gamma))$ –> $(\alpha$ –> $\gamma)$
(v) β –> $(\alpha$ –> $\beta)$
(vi) $(\alpha\&(\alpha$ –> $\beta))$ –> β
(vii) α –> $(\alpha|\beta)$
$(viii)$ $(\alpha|\beta)$ –> $(\beta|\alpha)$
(ix) $((\alpha$ –> $\beta)\&(\beta$ –> $\gamma))$ –> $((\alpha|\beta)$ –> $\gamma)$
(x) $(!\alpha)$ –> $(\alpha$ –> $\beta)$
(xi) $((\alpha$ –> $\beta)\&(\alpha$ –> $(!\beta)))$ –> $(!\alpha)$
(xii) $\alpha|(!\alpha)$

The system CL has one single rule of inference:

Principle 29 (Modus ponens) *From α and α –> β, β may be derived.*

The property of a sentence α of being a CL-theorem is denoted by $\underset{CL}{|\!\!-\!\!-} \alpha$. Following Heyting, the intuitionistic logic (IL) is the (CL) with the axiom (xii) omitted, and the same inference rule.

Theorem 73 *The following validity statements are equivalent:*

(i) $\underset{CL}{|\!\!-\!\!-} \alpha$.
(ii) $\models \alpha$.
(iii) *There exists a Boolean algebra B such that $B \models \alpha$.*
(iv) $BA \models \alpha$.

See [376] for details. We have this weaker relation:

Theorem 74 *The topos validity $\mathbf{C} \models \alpha$ implies classical validity $\underset{CL}{|\!\!-\!\!-} \alpha$.*

Definition 177 *A topos is Boolean iff the Heyting algebra $Sub(X)$ of each object X is Boolean.*

Theorem 75 *The following statements for a topos $\mathbf{C}$ are equivalent:*

(*i*) $\mathbf{C}$ *is Boolean.*
(*ii*) $Sub(\Omega)$ *is a Boolean algebra.*
(*iii*) $true : 1 \to \Omega$ *has a complement in* $Sub(\Omega)$.
(*iv*) $false : 1 \to \Omega$ *is the complement of* $true$ *in* $Sub(\Omega)$.
(*v*) $true \cup false = Id_\Omega$ *in* $Sub(\Omega)$.
(*vi*) $\mathbf{C}$ *is* classic, *i.e.,* $true \sqcup false : 1 \sqcup 1 \to \Omega$ *is iso.*
(*vii*) *The first inclusion* $\iota_1 : 1 \to 1 \sqcup 1$ *is a subobject classifier.*

For a proof, see [376, p.156 ff.].

Theorem 76 *If the topos* $\mathbf{C}$ *is Boolean, then* $\mathbf{C} \models \alpha | !\alpha$ *for all sentences* α.

Theorem 77 *For a topos* $\mathbf{C}$*, the following are equivalent:*

(*i*) $\mathbf{C} \models \alpha$ *iff* $\underset{CL}{|\!\!-\!\!-} \alpha$ *for all* α.
(*ii*) $\mathbf{C} \models \alpha | !\alpha$ *for all* α.
(*iii*) $Sub(1)$ *is a Boolean algebra.*

Theorem 78 *We have* $HA \models \alpha$ *iff* $\underset{IL}{|\!\!-\!\!-} \alpha$.

Theorem 79 *For all topoi* $\mathbf{C}$*, the validity* $\underset{IL}{|\!\!-\!\!-} \alpha$ *implies* $\mathbf{C} \models \alpha$.

G.5.2 Predicate Logic

Predicate calculus generates a richer set of sentences whose validity is a function of the interpretation of predicate variables and individual variables and not only of abstract propositional variables. We are given a set $\Upsilon = \{\iota_0, \iota_1, \ldots\}$ of *individual variables*, a set $\Xi = \{!, \&, |, ->, \exists, \forall\}$ of *predicate connectives*, a set $\Pi = \coprod_{i=0,1,2,\ldots} \Pi_i, \Pi_i = \{A^i, B^i, \ldots\}$ of *i-ary predicate variables*, and a set $\Delta = \{(,)\}$ of brackets as above. Within the free monoid $PEX = FM(\Upsilon \sqcup \Xi \sqcup \Pi \sqcup \Delta)$ of *predicate expressions*, one exhibits the subset $FO(PEX)$ of *formulae* as follows:

Property 6 *Given the symbols* $\Upsilon, \Xi, \Pi, \Delta$*, we require*

(*i*) (Atomic formulae) $A^i \iota_{i_1} \iota_{i_2} \ldots \iota_{i_i} \in FO(PEX)$ *for any* $A^i \in \Pi_i$ *and* $\iota_{i_k} \in \Upsilon$*, and* 0*-ary predicate variables are constants.*
(*ii*) (Propositional formulae) *If* $\alpha, \beta \in FO(PEX)$*, then*

$$(!\alpha) \in FO(PEX),$$
$$(\alpha \& \beta) \in FO(PEX),$$
$$(\alpha | \beta) \in FO(PEX),$$
$$(\alpha \; -> \beta) \in FO(PEX).$$

(*iii*) (Quantifier formulae) *If* $\alpha \in FO(PEX), x \in \Upsilon$*, then*

$$(\forall x)\alpha \in FO(PEX),$$
$$(\exists x)\alpha \in FO(PEX).$$

An individual variable x which appears in the formula after an expression of type $(\forall x)$ or $(\exists x)$ is *bound*, otherwise it is *free*.

A model $\mathfrak{M}$ of a predicate logic $FO(PEX)$ is a set M, together with i-ary relations $a^i, b^i \ldots \subseteq M^i$ (elements $a^0 \in M$ for constants). In such a model, the formal predicate expressions are interpreted via the interpretation of atomic formulae $A^i\iota_{i_1}\iota_{i_2}\ldots\iota_{i_i}$ by the truth value of $(x_{i_1}, x_{i_2}, \ldots x_{i_i}) \in a^i$. Whereas the interpretation of a quantifier formula $(\forall x)\alpha$ means "true" if the truth value of the interpretation of α is "true" for all valuations in M of all occurrences of the variable x, and the interpretation of a quantifier formula $(\exists x)\alpha$ means "true" if the truth value of the interpretation of α is "true" for at least one value in M of the occurrences of the variable x—of course, this is only decided if no free variables are left, in which case one calls the formula a *sentence*, otherwise, no truth value is defined and the formula is just a truth-value-function of the left free variables. We write $\mathfrak{M} \vDash \alpha[x]$ for truth value "true" for the evaluation $[x]$ of the free variables of α.

The recursive calculation of truth values of compound formulas relates to the Boolean algebra $Sub(M^m)$ as follows: To begin with, if a logical combination of two formulae is considered, one may suppose that both variable sets of these formulas coincide by just taking their union if they do not coincide. If we fix such a variable set of cardinality m, say, the truth evaluation of a formula α with (at most) these m free variables can be described by the inverse image $supp(\alpha) \subseteq M^m$ of *true*. Then evidently, $supp(!\alpha) = M^m - supp(\alpha)$, $supp((\alpha\&\beta)) = supp(\alpha) \cap supp(\beta)$, $supp((\alpha|\beta)) = supp(\alpha) \cup supp(\beta)$, $supp((\alpha -> \beta)) = supp(!\alpha) \cup supp(\beta)$. For the quantifiers, we have this situation: If a variable x is bound by a quantifier, we have the support $supp(\alpha) \subseteq M^m$ of the given formula α and the support $supp((\forall x)\alpha) \subseteq M^{m-1}$ or $supp((\exists x)\alpha) \subseteq M^{m-1}$, respectively. Suppose that x is the i^{th} variable, then we have the projection $p_i : M^m \to M^{m-1}$, which omits the i^{th} coordinate, and the inverse image map $p^* : 2^{M^{m-1}} \to 2^{M^m}$. If S is a support of a formula α in M^m, then the support $(\forall x)(S)$ of $(\forall x)\alpha$ is the set $\{(y_1, \ldots, y_{m-1})|(y_1, \ldots y_{i-1}, x, y_{i+1}, \ldots y_{m-1}) \in S \text{ for all } x \in M\}$, while the support $(\exists x)(S)$ of $(\exists x)\alpha$ is the set $\{(y_1, \ldots y_{m-1})|(y_1, \ldots y_{i-1}, x, y_{i+1}, \ldots y_{m-1}) \in S \text{ for at least one } x \in M\}$.

Proposition 126 *The functor of partially ordered sets $\forall x : 2^{M^m} \to 2^{M^{m-1}}$ is a right adjoint of $p^* : 2^{M^{m-1}} \to 2^{M^m}$, while $\exists x : 2^{M^m} \to 2^{M^{m-1}}$ is its left adjoint, in other words, $p^*(X) \subset Y$ iff $X \subset \forall(x)(Y)$ and $Y \subset p^*(X)$ iff $\exists(x)(Y) \subset X$.*

The topos-theoretic generalization of this result is the following theorem; for a proof, see [639, p.209,p.206].

Theorem 80 *If $f : A \to B$ is a morphism in the topos $\mathbf{C}$, then the functor $Sub(B) \to Sub(A)$ of Heyting algebras (which are viewed as categories via their partial orders as morphisms) associated with the natural morphism $\Omega^f : \Omega^B \to \Omega^A$ has a right adjoint functor $\forall_x$ and a left adjoint functor $\exists_x$.*

In order to rewrite the predicate calculus in general topoi, one uses the characteristic maps associated with supports of predicates as follows: If M is a non-zero object of the topos $\mathbf{C}$, we consider the characters $\chi_{a^m} : M^m \to \Omega$ of the "supports" $a^m \subset M^m$ of m-ary predicates. Their recombination via logical constructions runs as follows: Using the morphisms of negation, conjunction, disjunction, and implication defined above in Section G.5.1 for Ω, one has the evident combination of supports of formulas via their characters. The new thing here is the definition of quantifier supports. Given a character $\chi_\alpha : M^m \to \Omega$, we have the adjoint morphism $ad_i(\chi_\alpha) : M^{m-1} \to \Omega^M$ with respect to the i^{th} coordinate. Then we have two arrows $\forall_M, \exists_M : \Omega^M \to \Omega$. The first $\forall_M$ is the character of the adjoint of the composite $true\circ! \circ pr_M : 1 \times M \to M \to \Omega$. The second $\exists_M$ is the character of the image of the composed map $pr_{\Omega^M} \circ \varepsilon_M : \varepsilon \rightarrowtail \Omega^M \times M \to \Omega^M$, where $\varepsilon_M : \varepsilon \rightarrowtail \Omega^M$ is the subobject whose character is the evaluation map $ev_M : \Omega^M \times M \to \Omega$ adjoint to the identity on Ω^M. We then have these formulas:

$$\forall(x)\alpha \text{ has the character } \forall_M \circ ad_i(\chi_\alpha),$$
$$\exists(x)\alpha \text{ has the character } \exists_M \circ ad_i(\chi_\alpha).$$

G.5.3 A Formal Setup for Consistent Domains of Forms

Since forms do not automatically exist if we allow circularity, it is important to set up a formal mathematical context in order to describe what a logically consistent domain of forms should be. This mathematical formalism turns out to be valid in an interesting general context. We have been working in the topos $\mathbf{Mod}^{@}$ of presheaves over the category **Mod**, where we have the Yoneda embedding $Y : \mathbf{Mod} \to \mathbf{Mod}^{@}$. Without loss of generality, we may identify **Mod** with the full subcategory of represented presheaves $@M, M$ a module.

More generally, we may consider *Yoneda pairs* $\mathcal{R} \subset \mathcal{E}$, where $\mathcal{R}$ is a full subcategory of a topos $\mathcal{E}$, $\mathcal{R}$ playing the role of represented modules (we also say that $\mathcal{R}$ is a *Yoneda subcategory*). This means that we require that the canonical Yoneda functor $\mathcal{E} \to \mathcal{E}^{@} \to \mathcal{R}^{@}$ be fully faithful. By Yoneda's Lemma, we may identify the evaluation $M@F$ of a "presheaf" $F \in \mathcal{E}$ at a "module" $M \in \mathcal{R}$ by the morphism set from M to F: $M@F = Hom_{\mathcal{E}}(M, F)$. This setup in particular includes the classical case of $\mathcal{E} = \mathbf{Ens}$ and $\mathcal{R}$ the one-element category consisting of a singleton 1 (the terminal object in **Ens**), say $1 = \{\varnothing\}$, and its unique identity morphism. In this case, we may identify $1@F$ and the set F.

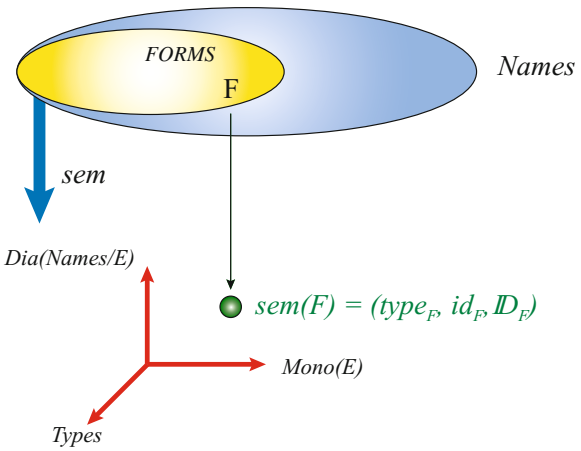

Fig. G.1. The formal setup of a semiotic of $\mathcal{E}$-forms.

To achieve the intended formalism, we consider the set $Mono(\mathcal{E})$ of monomorphisms in $\mathcal{E}$. We further consider the set

$$Types = \{\mathbf{Simple}, \mathbf{Syn}, \mathbf{Limit}, \mathbf{Colimit}, \mathbf{Power}\}$$

of form types. And we need the free monoid $Names = FM(UNICODE)$ over the $UNICODE$ alphabet (which is an extension of the $ASCII$ alphabet to non-European letters). We next need the set $Dia(Names)$ of all diagram schemes with vertexes in $Names$. More precisely, a diagram scheme over $Names$ is a finite directed multigraph whose vertexes are the elements of a subset of $Names$, and whose arrows $i : A \to B$ are triples (i, A, B), with $i = 1, \ldots$ natural numbers to identify arrows for given vertexes.

Next, consider the set $Dia(Names/\mathcal{E})$ of diagrams on $Dia(Names)$ with values in $\mathcal{E}$. Such a diagram is a map

$$dia : D \to \mathcal{E}$$

which to every vertex of the diagram scheme D associates an object of $\mathcal{E}$ and to every arrow associates a morphism in $\mathcal{E}$ between corresponding vertex objects. So $i : A \to B$ is mapped to the morphism $dia(i) : dia(A) \to dia(B)$. We also will identify two such diagrams iff their arrows for given names A, B are permutations of each other, i.e., we only consider the orbits of diagrams modulo the permutation group of arrows on given names. Why? Because any construction of limits or colimits is invariant under this group since the limit condition is a logical conjunction which does not depend on the numbering of the arrows.

So this identification will always be valid unless explicitly suspended. Observe further that a multiple appearance of a vertex in a diagram scheme is not allowed, so when constructing diagram schemes upon form names, one must add synonymous forms when multiple appearance of one and the same form in a diagram is desired. This is the advantage of form names: the annoying indexing of mathematical names can be absorbed by intrinsic renaming on the level of form names.

With these notation, we can define a *semiotic of $\mathcal{E}$-forms* as follows (see also Figure G.1):

Definition 178 *A* semiotic of $\mathcal{E}$-forms *is a set map*

$$sem : FORMS \to Types \times Mono(\mathcal{E}) \times Dia(Names/\mathcal{E})$$

defined on a subset $FORMS \subset Names$ with the following properties (i) to (iii)*. To ease language, we use the following notation and terminology:*

- *An element $F \in FORMS$ is called a* form name*, and the pair (F, sem) a* form*,*
- *$pr_1 \cdot sem(F) = t(F)$ (=type of F),*
- *$pr_2 \cdot sem(F) = id(F)$ (= identifier of F),*
- *$domain(id(F)) = fun(F)$ (= topor[12] or "space" of F),*
- *$codomain(id(F)) = frame(F)$ (= frame or "frame space" of F),*
- *$pr_3 \cdot sem(F) = coord(F)$ (= coordinator of F).*

Then these properties are required:

(*i*) *The empty word $\varnothing$ is not a member of $FORMS$*

(*ii*) *For any vertex X of the coordinator diagram $coord(F)$, if $X \in FORMS$, then we have*

$$coord(F)(X) = fun(X).$$

(*iii*) *If the type $t(F)$ is given, we have the following for the corresponding frames:*

- *For* **Syn** *and* **Power***, the coordinator has one vertex $G \in FORMS$ and no arrows, i.e., $coord(F) : G \to fun(G)$, which means that in these cases, the coordinator is determined by a form name G. Further, for* **Syn***, we have $frame(F) = fun(G)$, and for* **Power***, we have $frame(F) = \Omega^{fun(G)}$, if $coord(F) : G \to fun(G)$, as above.*
- *For* **Limit** *and* **Colimit***, the coordinator is any diagram $coord(F)$ whose names are all in $FORMS$. Further, for* **Limit***, we have $frame(F) = lim(coord(F))$, and for* **Colimit***, we have $frame(F) = colim(coord(F))$.*
- *For type* **Simple***, the coordinator has the unique vertex $\varnothing$, and a value $coord(F) : \varnothing \to M$ for a 'module' $M \in \mathcal{R}$ (i.e., a represented presheaf $M = @X$ in the case of presheaves over* **Mod***), or, in a more sloppy notation: $coord(F) = M$.*

Here, circular forms are evidently included via form names which refer to themselves in their diagrams or in deeper recursion structures. With this definition we may discuss the existence and size of form semiotics, i.e., the extent of the $FORMS$ set, maximal such sets, gluing such sets together along compatible intersections, etc. However, we shall not pursue this interesting and logically essential branch for simple reasons of space and time.

G.5.3.1 Morphisms Between Semiotics of Forms

Although the theory of form semiotics is in its very beginnings, it is clear that two form semiotics with intersecting domains $FORM_1$ and $FORM_2$ nee not be contradictory even if the semiotic maps do not coincide on the intersection $FORM_1 \cap FORM_2$. In fact, it could happen that on this intersection, the maps are just "equivalent" semiotics. More generally, it could happen that two form semiotics have subsemiotics

[12] The functor in the special case $\mathcal{E} = \mathbf{Mod}^{@}$.

which are in complete correspondence and therefore we may glue them to a global semiotic structure. In other words: It is reasonable and feasible to consider morphisms and then categories of form semiotics and therefore isomorphisms of semiotics, which enables us to construct global semiotics just by gluing together local "charts" as usual.

Let us abbreviate $Sema(\mathcal{E}) = Types \times Mono(\mathcal{E}) \times Dia(Names/\mathcal{E})$, *Sema* being an abbreviation for semantic target space. Suppose that we are given two form semiotics $sem_1 : FORMS_1 \to Sema(\mathcal{E}_1), sem_2 : FORMS_2 \to (\mathcal{E}_2)$. We correspondingly denote by

$$fun_1, fun_2, t_1, t_2, id_1, id_2, frame_1, frame_2, \text{ and } coord_1, coord_2$$

the respective maps. Consider pairs (u, v) where $u : FORMS_1 \to FORMS_2$ is a set map, and where $v : \mathcal{E}_1 \to \mathcal{E}_2$ is a logical functor (see Appendix G.3) sending $\mathcal{R}_1$ to $\mathcal{R}_2$. We say that the pair (u, v) *is morphic* (for $FORMS_1, FORMS_2$) iff

1. We have $u(\varnothing) = \varnothing$.
2. The functors commute with u, v, i.e., we have $v \cdot fun_1 = fun_2 \cdot u$.
3. The type is invariant under u, i.e., $t_2 \cdot u = t_1$.

In particular, mono- and epimorphisms on $\mathcal{E}_1$ are preserved (see Appendix G.2.2). Suppose that we are given a diagram scheme

$$C = coord_1(F) : D \to \mathcal{E}_1$$

(modulo permutations on the numberings of the arrows between fixed names, as announced!) associated with the form name $F \in FORM_1$. Let $|D|$ be the vertex names of D. We define a diagram E as follows. Its vertexes are the image $|E| = u(|D|)$. For every vertex pair (X, Y) of $|E|$ we take all arrows $i : A \to B$ with $X = u(A), Y = u(B)$. By lexicographic order on the triples (A, B, i), we can order all these arrows and index them with positive natural numbers $j = 1, \ldots n(X, Y)$. This defines a unique new diagram scheme. Secondly, we define a new diagram $C' : E \to \mathcal{E}_2$ as follows: If the arrow $i : A \to B$ gives arrow $j(i) : X \to Y$, the new diagram C' maps this arrow to the morphism

$$v(C(i)) : fun_2(X) = v(fun_1(A)) \to v(fun_1(B)) = fun_2(Y).$$

Denote this diagram by $(u, v)(C)$. Clearly, since we only retain orbits of diagram schemes, we have functoriality, i.e., if $(u_1, v_1), (u_2, v_2)$ are two such morphic pairs for $FORMS_1, FORMS_2$, and $FORMS_2, FORMS_3$, respectively we have

$$(u_2, v_2)((u_1, v_1)(C)) = (u_2 \cdot u_1, v_2 \cdot v_1)(C). \tag{G.28}$$

Definition 179 *A morphic pair (u, v) for the pair $FORMS_1, FORMS_2$ is said to be a* morphism of form semiotics $(u, v) : sem_1 \to sem_2$ *if the following semiotic data with each given form name $F \in FORMS_1$ are verified:*

1. *Let F be simple, i.e., $F \underset{Id}{\longrightarrow} \mathbf{Simple}(M)$. Then $u(F) \underset{Id'}{\longrightarrow} \mathbf{Simple}(N)$[13], and we require that*
   - *$N = v(M)$,*
   - *Id'=v(Id), i.e., the monomorphism of Id' is the v-image of the monomorphism of Id—the domains and codomains are already the right ones, only the morphism (a monomorphism by the conservation of limits) has to fit—so*

$$Id' = v(Id) : fun_2(u(F)) = v(fun_1(F)) \rightarrowtail N = v(M).$$

2. *Let F be synonymous, i.e., $F \underset{Id}{\longrightarrow} \mathbf{Syn}(G)$. Then $u(F) \underset{Id'}{\longrightarrow} \mathbf{Syn}(G')$, and we require that $G' = u(G)$ and $Id' = v(Id) = fun_2(u(F)) \rightarrowtail fun_2(u(G))$.*

[13] Observe that in the case $\mathcal{R} = \mathbf{Mod}, \mathcal{E} = \mathbf{Mod}^{@}$ of presheaves, we usually write the module as a coordinator, but we mean its represented functor $@M$.

3. *Let F be of power type, i.e., $F \underset{Id}{\longrightarrow} \mathbf{Power}(G)$. Then $u(F) \underset{Id'}{\longrightarrow} \mathbf{Power}(G')$, and we require that $G' = u(G)$ and $Id' = v(Id) = fun_2(u(F)) \rightarrowtail v(\Omega^{fun_1(G)}) \xrightarrow{\sim} \Omega^{v(fun_1(G)} = \Omega^{fun_2(u(G))}$.*
4. *Let F be of limit (resp. colimit) type, i.e., $F \underset{Id}{\longrightarrow} \mathbf{Limit}(C)$ (resp. $F \underset{Id}{\longrightarrow} \mathbf{Colimit}(C)$) with $C = coord(F)$. Then we have $u(F) \underset{Id'}{\longrightarrow} \mathbf{Limit}(C')$ (resp. $u(F) \underset{Id'}{\longrightarrow} \mathbf{Colimit}(C')$). We then require that $C' = (u, v)(C)$ and that $Id' = v(Id) = fun_2(u(F)) = v(fun_1(F))$ $\rightarrowtail v(lim(C)) \xrightarrow{\sim} lim(v \cdot C) \xrightarrow{\sim} lim((u, v)(C))$ (resp. the analogous expression with colimits).*

In any case, the associated form $u(F)$ is related to its ingredients through the given functor v and the recursive constructions on the coordinators via u.

Clearly, the evident composition of two morphisms from formula (G.28) is again a morphism, and we obtain the category *ForSem* of form semiotics.

G.5.3.2 Local and Global Form Semiotics

It is clear what one should understand by a *global form semiotic*: This is a set G, together with a covering I and an atlas $f_i : I_i \xrightarrow{\sim} FORMS_i$ of bijections onto domains of form semiotics $semi_i : FORMS_i \to Sema$ such that all the induced bijections $u_{i,j} : FORMS_i|_j \xrightarrow{\sim} FORMS_j|_i$ extend to isomorphisms $(u_{i,j}, v_{i,j})$ of form semiotics. This means in particular that all intersections $FORMS_i|_j$ are form domains of form sub-semiotics in sem_i, and that the underlying functors on $\mathcal{E}_i$ are compatible. We leave the details to the interested reader.

G.5.3.3 Connotator Form Semiotics

Denotator and form names were very simple word objects in the previous setup. But name spaces may also be required to encompass more articulated structures, in other words: we want names to be denotators as well, thereby turning the denotator concept into a 'connotator' concept. Here is the formal setup.

We again suppose given a Yoneda pair $\mathcal{R}, \mathcal{E}$. We also retain the set $Mono(\mathcal{E})$. We are given two sets $\mathcal{D}$ of denotators and $\mathcal{F}$ of forms, they are supposed to parametrize denotators and forms according to the following system of maps. We have three maps on $\mathcal{D}$:

$$\begin{aligned} &coordinate : \mathcal{D} \to \mathcal{E}, \\ &form : \mathcal{D} \to \mathcal{F}, \\ &denotatorName : \mathcal{D} \to \mathcal{D}. \end{aligned}$$

The coordinate C of a denotator is supposed to be any morphism with domain A within $\mathcal{R}$, which is called the denotator's address. We require that a denotator be uniquely determined by its coordinate C, form F, and denotatorName N. This is why denotators are also written as quadruples $N : A@F(C)$, where the address is denoted for comfort since it is important information.

The denotator's form mimics the space where the denotator lives. To this end, we need two more sets. The set of types is $\mathcal{T} = \{\mathbf{Limit}, \mathbf{Colimit}, \mathbf{Power}, \mathbf{Simple}\}$, it contains the basic constructors of objects in a topos. But we omit synonymy in this generic setup because it can be mimicked by a limit with just one vertex. We also need the set $Diagrams(\mathcal{D}/\mathcal{E})$ of finite diagrams whose vertexes are denotators, and whose arrows are numbered by $1, 2, 3, \ldots$ as above. This means that the diagram schemes are these symbols, and that the evaluation of the diagram scheme yields objects and morphisms in $\mathcal{E}$. Forms have, by hypothesis, uniquely determined values under these four maps:

$$\begin{aligned} &formName : \mathcal{F} \to \mathcal{D}, \\ &identifier : \mathcal{F} \to Mono(\mathcal{E}), \\ &diagram : \mathcal{F} \to Diagrams(\mathcal{D}, \mathcal{E}), \\ &type : \mathcal{F} \to \mathcal{T}. \end{aligned}$$

This means that a form can be written as $FN : Id.T(Dg)$, where FN is the form's name denotator, Id its identifier, T its type, and Dg its diagram.

We impose a small number of axioms for these structures. To this end, we call the domain $dom(Id)$ of a form $FN : Id.T(Dg)$ the form's *space*, whereas the codomain $cod(Id)$ is called its *frame space*. Accordingly, for a denotator $N : A@F(C)$, $codom(C) = dom(Id)$, and the composition $Id \circ C$ with its form's identifier Id is called the *frame coordinate*, it uniquely determines the denotator's coordinate.

Axiom 8 *Here are the conditions for this setup:*

(*i*) *The map formName is injective, i.e., the form's name is a key.*
(*ii*) *For all form diagrams, except for simple type, the vertex denotators of the diagram schemes are form names, and their values are the spaces of the respective forms.*
(*iii*) *If the form's type is* **Limit** *or* **Colimit***, its frame space is the limit or colimit of the diagram.*
(*iv*) *If the form's type is* **Power***, the diagram has just one vertex and no arrows, and the frame space is Ω^S, where S is the space of the vertex form.*
(*v*) *There is a denotator $\varnothing$ which is not a form name, and for a simple type form, the diagram has exactly the vertex $\varnothing$, no arrow, and the value is a 'representable' object X in $\mathcal{R}$. In other words, the simple type frame space is just a representable object in disguise. Such a diagram is represented by $\varnothing X$.*

The language of forms and denotators has been encoded in an ASCII-based textual form, like TEX, which is therefore called *Denotex* and is available in BNF[14]. In RUBATO®, a Denotex parser is available for communication with Denotex files. Our present notation in this section, such as $FN : Id.T(Dg)$ for forms and $N : A@F(C)$ for denotators, is an illustration of the Denotex notation.

Example 130 An elementary form for names can be set up as follows: The form NF represented by $fn : Id.\mathbf{Simple}(Dg)$ is simple with the diagram $\varnothing\mathbb{Z}\langle UNICODE\rangle$. The identifier Id is the identity on the representable presheaf $@\mathbb{Z}\langle UNICODE\rangle$, and the name fn is a denotator $fn : 0@FN(C)$, whose coordinate C is the zero-addressed homomorphism $C : 0 \to \mathbb{Z}\langle UNICODE\rangle$ with value $C(0) =$ "NameForm" with $denotatorName(fn : 0@NF(C)) = fn$, i.e., it is its proper name denotator. So its identification resides on its coordinate value "NameForm" and the form named fn. This identifies the entire NF form. Then, general $UNICODE$ names n may be defined by $n : 0@NF(Cn)$, where the value $Cn(0) =$ "anyName" is any $UNICODE$ string combination, such as "3.Violin+4.Piano", and which are their proper name denotators, i.e., $denotatorName(n : 0@NF(C)) = n$.

[14] Denotex was developed in collaboration with Thomas Noll, Jörg Garbers, Stefan Göller, and Stefan Müller.

H

Complements on General and Algebraic Topology

H.1 Topology

Refer to [527] for general topology, and to [993] for algebraic topology.

H.1.1 General

A topological space is a pair $(X, Open_X)$ of a set X and a set $Open_X$ of *open* subsets of X such that X is open, $U \cap V$ is open if U, V are so, and $\bigcup_i U_i$ is open for any family (U_i) of open sets, in particular the union of the empty family, the empty set, is open. The complement $X - U$ of an open set U is called *closed*. Therefore the collection $Closed_X$ of closed sets fits with the corresponding axioms: the union of any two closed sets is closed, the intersection of any family of closed sets is closed[1], and the empty set is closed. If we define the *closure* $\overline{Y}$ of any subset of X as the intersection of all closed sets containing Y, then the topology is again defined by the axioms for the Kuratowski closure operator $: 2^X \to 2^X$, i.e., $\overline{\varnothing} = \varnothing$, is idempotent, $Y \subseteq \overline{Y}$, and $\overline{Y \cup Z} = \overline{Y} \cup \overline{Z}$.

Given two topologies $Open_X, Open'_X$, one says that $Open_X$ *is coarser than* $Open'_X$ or that $Open'_X$ *is finer than* $Open_X$ iff $Open_X \subseteq Open'_X$. On any set X, the coarsest topology consists just of X and of the empty set, it is called the *indiscrete topology*, whereas the finest topology is the powerset of X, it is called the *discrete topology*. The intersection of any family of topologies on X is the finest topology which is coarser than each member of the family. Every set of subsets $\mathcal{S}$ of X is contained in the intersection of all topologies containing this subset, a family containing at least the discrete topology. It consists of all unions of finite intersections (the empty intersection gives X) of members of $\mathcal{S}$ and is denoted by $Open(\mathcal{S})$.

A *neighborhood* W of $x \in X$ is a subset containing an open set U which contains x. Finite intersections of neighborhoods of x are neighborhoods, supersets of neighborhoods are neighborhoods. An *accumulation point* of a subset Y of X is a point not in Y which intersects Y in each of its neighborhoods. The closure of a subset Y of X is the union of Y and of its accumulation points. The *interior* U^o of a subset is the union of all open subsets of U. It is also the complement of the closure of its complement. The interior operator has an evident set of axioms corresponding to the closure axioms which also characterize the topology. The *boundary* ∂U of a subset is the difference $\overline{U} - U^o$.

A subset $\mathcal{B}$ of open sets is called a *base for the topology* iff any open set is the union of a family of $\mathcal{B}$ members, or, equivalently, every neighborhood of a point contains a neighborhood from $\mathcal{B}$. The axioms for a set of subsets $\mathcal{B}$ of X to be a base for a topology is that $X = \bigcup \mathcal{B}$, and that for any two $U, B \in \mathcal{B}$, $U \cap V$ is the union of members of $\mathcal{B}$. A *subbase for a topology* $Open_X$ on X is a set $\mathcal{S}$ of subsets of X such that $Open_X = Open(\mathcal{S})$.

[1] The intersection of the empty family being defined as the total space X.

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G. Mazzola, *The Topos of Music IV: Roots*, Computational Music Science,
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H.1.2 The Category of Topological Spaces

Suppose that $(X, Open_X), (Y, Open_Y)$ are topological spaces. A set map $X \to Y$ is *continuous* iff the inverse map $2^Y \to 2^X$ induces a map $Open_Y \to Open_X$. The set-theoretic composition of continuous maps is continuous, the identity map is so, and therefore, we have the *category* **Top** *of topological spaces and continuous maps*. An isomorphism of topological spaces is called a *homeomorphism*. Any subset W of a topological space becomes a topological space by the coarsest topology $Open_W = Open_X|W$ such that the inclusion $W \subset X$ is continuous; its open sets are just the intersections of open sets of X with W, this topology is called the *relative topology on* W. More generally, given any set map $f : X \to Y$ into a topological space $(Y, Open_Y)$ the coarsest topology $Open_Y|f$ (smallest set of open sets) on X such that f becomes continuous is given by the set of inverse images of open sets of Y, we also call it the *relative topology with respect to* f. Conversely, for a set map $f : X \to Y$, where $(X, Open_X)$ is a topological space, we have a finest topology such that f becomes continuous, it is given by the set of all subsets of Y such that their inverse image is open in X. This is the *quotient topology* $Open_X/f$. If (X_i) is a family of topological spaces, the cartesian product $\prod_i X_i$ has the coarsest topology such that the projections to all factors become continuous. A base of this *product topology* is given by the products $\prod_i U_i$ of open sets $U_i \subseteq X_i$ with $U_i = X_i$ except for a finite number of indices. This is a limit in the category **Top**. The coarsest topology on the set-theoretical limit $lim()$ of a diagram of continuous maps is the limit in **Top**, a similar construction (this time with the finest topology) yields the colimit of a diagram of continuous maps.

If we are given a family $f_i : X_i \to X$ of set maps whose domains are topological spaces, there is a finest topology which makes these maps continuous. Its universal property is that with this topology on X, a map $g : X \to Y$ into a topological space Y is continuous iff all compositions $g \circ f_i$ are so. This is a particular case of a quotient topology for the situation $\coprod_i X_i \to X$. This topology is called the *coinduced topology*. If the maps f_i are inclusions of subspaces X_i of a topological space X, the topology of X is called *coherent* or *weak* if it is coinduced from the relative topologies on the spaces X_i.

If we are given a set X, together with a collection of subsets C_i of X which are topological spaces such that for all indexes i, j, the intersections $C_i \cap C_j$ have the same relative topology as inherited from C_i or from C_j, and that these intersections are closed in both, C_i and C_j. Then the coinduced topology is coherent with this family, in other words, the coinduced topology relativizes to the given topologies on all C_i.

H.1.3 Uniform Spaces

Topologies are often defined by relations that stem from metrical distance functions. The axiomatics is as follows:

Definition 180 *A* uniformity *on a set* X *is a set* $\mathcal{U}$ *of* uniform sets $U \subseteq X^2$ *such that:*

(*i*) *Each uniform set contains the diagonal* Δ.
(*ii*) *If* U *is uniform, so is* U^{-1}.
(*iii*) *If* U *is uniform, then there is a uniform* V *such that* $V \circ V \subset U$.
(*iv*) *If* U, V *are uniform, then so is* $U \cap V$.
(*v*) *If* U *is uniform, then so is every superset in* X^2.

The prototype of a uniformity is given by a distance function, i.e., a pseudo-metric $d : X \times X \to \mathbb{R}$ as defined in definition 185 in Appendix I.1.1. The uniformity contains all $U \subseteq X^2$ which contain a set of type $U_\epsilon = \{(x, y)|d(x, y) < \epsilon\}, \epsilon > 0$.

Each uniformity $\mathcal{U}$ gives rise to a *uniform topology* $Open(\mathcal{U})$ whose open sets are those V such that for each $x \in V$, there is a uniform set U with $U[x] \subset V$, where $U[x] = \{y|(x, y) \in U\}$. So the uniform topology imitates metrical neighborhoods.

H.1.4 Special Issues

Definition 181 *A topological space X is said to be:*

(i) T_0 *iff for any two different points $x, y \in X$, at least one of them is not the specialization of the other;*
(ii) T_1 *iff every point is closed, i.e., no other point dominates it;*
(iii) T_2 *(Hausdorff) iff every two different points have disjoint neighborhoods.*

Definition 182 *A subset $L \subset X$ of a topological space X is said to be* locally closed *iff one of the equivalent properties holds:*

(i) $L = O \cap C$, O *open,* C *closed.*
(ii) Every point $l \in L$ has an open neighborhood U_l such that $U_l \cap L$ is closed in U_l.
(iii) L *is open in its closure in* X.

See [147, I,§3.3] for a proof.

Definition 183 *A topological space X is called* quasi-compact *iff every covering of X by open sets admits a finite subcovering. A Hausdorff quasi-compact space is called* compact.

Typically, prime spectra of commutative rings are quasi-compact but not compact.

H.2 Algebraic Topology

Refer to [993] for this section.

H.2.1 Simplicial Complexes

A *simplicial complex* K is a set V of *vertexes*, together with a subset K of 2^V whose elements are called *simplexes* such that (1) each singleton $\{v\}, v \in V$ is a simplex, (2) each non-empty subset of a simplex is a simplex. If for a simplex s of K, $card(s) = q+1$, one says that s is a *q-simplex* or a *q-dimensional simplex*. A subsimplex $s' \subseteq s$ of a simplex s is called a *face of s*; it is called a *q-face* if it is a q-dimensional simplex, we also write $s' \leqslant s$ instead of $s' \subseteq s$. Evidently, a simplicial complex is completely determined by its simplex set K and may be identified with it.

Example 131 Let U be a covering of a set X by non-empty subsets. The *nerve* $n(U)$ of U is the simplicial complex with $V = U$, and the simplexes s being those finite sets $s = \{u_0, u_1, \dots u_p\}$ in U which have non-empty intersection $\cap s = \bigcap_i u_i$.

The *dimension* $dim(K)$ of a simplicial complex K is the maximal dimension of its simplexes, including the special cases $dim(\varnothing) = -1$ $dim(K) = \infty$ if no maximal dimension exists. A simplicial map $f : K_1 \to K_2$ is a set map $f : V_1 \to V_2$ on the underlying vertex sets such that the induced map $2^f : 2^{V_1} \to 2^{V_2}$ carries simplexes to simplexes, i.e., restricts to a map $f : K_1 \to K_2$, meaning that if $s \in K_1$, then $f(s) \in K_2$. One may also say that it is a set map $F : K_1 \to K_2$ which is induced by a map f on the underlying vertex sets. The simplicial complexes and their simplicial maps define the *category* **Simpl** *of simplicial complexes.*

A *subcomplex* L of a simplicial complex K is a subset of simplexes which is also a simplicial complex. L is *full* iff a simplex of K whose vertexes belong to L is also in L. For example, given a simplicial complex K and a natural number k, the k-dimensional skeleton $K|k$ is the subcomplex of all simplexes of dimension $\leqslant k$. For a covering U, the k-dimensional skeleton of its nerve is denoted by $n_k(U)$.

Example 132 Let *Covens* be the category of set coverings, whose objects are pairs (X, I) of sets X and coverings I of X by non-empty subsets. The morphisms are pairs $(f, \phi) : (X, I) \to (Y, J)$ with $f : X \to Y, \phi : I \to J$ two maps such that for all $i \in I$, $f(i) \subset \phi(i)$. We then have the nerve functor $n : Covens \to$ **Simpl** : $(X, I) \mapsto n(I)$.

H.2.2 Geometric Realization of a Simplicial Complex

We have a functorially defined geometric representation of simplicial complexes K by topological spaces $|K|$ as follows. The set $|K|$ is the subset of those functions $\alpha : V(K) \to I = [0,1]$ into the real unit interval I such that

1. the support $supp(\alpha) = \{v \in V(K) | \alpha(v) \neq 0\}$ of α is a simplex,
2. $\sum_{v \in V(K)} \alpha(v) = 1$.

The value $\alpha(v)$ is called the v^{th} *barycentric coordinate* of α. On the set $I^{(V(K))}$ of functions with finite support, one has the Euclidean metric $d(\alpha, \beta) = \|\alpha - \beta\|_2$. We induce this metric and its associated topology (see Section H.1.3) on $|K|$ and denote it by $|K|_d$.

For a simplex $s \in K$, the closed simplex $|s|$ is defined by

$$|s| = \{\alpha \in K | supp(\alpha) \subset s\}.$$

Evidently, if $dim(s) = q$, there is a homeomorphism $|s|_d \xrightarrow{\sim} \Delta^q = \{x \in I^{q+1} | \sum x_i = 1\}$ onto the "standard closed q-simplex". If $s, t \in K$, either $s \cap t = \varnothing$ or a common face, and then $|s \cap t| = |s| \cap |t|$, so $|s|_d \cap |t|_d$ is closed in both, $|s|_d, |t|_d$, and the relative topologies from $|s|_d, |t|_d$ coincide on the intersection. By the remarks on coinduced topologies in Section H.1.2, we have the *coherent topology on* $|K|$ which is coinduced from the topologies on the closed simplexes. This means that

Fact 23 *A subset $E \subseteq |K|$ is closed/open iff each intersection $E \cap |s|_d$ is closed/open.*

Therefore, a function $f : |K| \to X$ into a topological space X is continuous iff its restrictions $f||s|$ are so for all simplexes s of K. In particular, the identity $|K| \to |K|_d$ is continuous, therefore, $|K|$ is Hausdorff, it is also normal, see [993, 3.1, Th.17]. Also, $|K|$ is compact iff K is finite. Call K *locally finite*, iff every vertex belongs to a finite number of simplexes. Then

Theorem 81 *For a simplicial complex K the following statements are equivalent:*

(*i*) *K is locally finite.*
(*ii*) *The identity $|K| \to |K|_d$ is a homeomorphism.*
(*iii*) *$|K|$ is metrizable, i.e., there is a metric whose topology is the coherent topology.*

See [993, 3.2, Th.8] for a proof.

If $f : K_1 \to K_2$ is a simplicial map, we have the continuous map

$$|f|(\alpha)(v) = \sum_{f(w)=v} \alpha(w)$$

which is continuous for both topologies on $|K|$. We are therefore given two functors $|?|, |?|_d : \mathbf{Simpl} \to \mathbf{Top}$ and a natural transformation $Id : |?| \to |?|_d$.

A continuous map $f : |K| \to X \subset \mathbb{R}^n$ is said to be *linear* iff $f(\alpha) = \sum_{v \in V(K)} \alpha(v) f(v)$ for all $\alpha \in |K|$. Any function on the vertexes may uniquely be extended to a continuous linear map, this is the universal property of affine pointsets in general position. In particular, the map $|f|$ associated with a simplicial map f is linear.

Definition 184 *A* geometric realization *of a simplicial complex K in $\mathbb{R}^n$ is a linear embedding* (*injection*) *of $|K|$ in $\mathbb{R}^n$.*

Theorem 82 *If a simplicial complex K has a geometric realization in $\mathbb{R}^n$, then it is countable, locally finite and has dimension $\leqslant n$. Conversely, if it is countable, locally finite, and has dimension $\leqslant n$, then it has a geometric realization as a closed subset of $\mathbb{R}^{2n+1}$.*

Example 133 For the nerve $n(U)$ of a finite covering U, we write $N(U)$ for the geometric realization $|n(U)|$, we also write $N_k(U)$ for $|n_k(U)|$.

H.2.3 Contiguity

A *simplicial pair* is a couple (K, L), where K is a subcomplex of L. A simplicial map of pairs $f : (K_1, L_1) \to (K_2, L_2)$ is a simplicial map $f : L_1 \to L_2$ which induces a simplicial map on the respective subcomplexes. Two simplicial maps $f, f' : (K_1, L_1) \to (K_2, L_2)$ are called *contiguous* if for every simplex s in K_1 or L_1, the union $f(s) \cup f'(s)$ is a simplex in K_2 or L_s. Contiguity is an equivalence relation and defines *contiguity classes* of simplicial maps.

Two continuous maps $f, g : X \rightrightarrows Y$ of topological spaces are called *homotopic* iff there is a continuous map (a homotopy) $F : X \times I \to Y$ such that $f = F(?, 0), g = F(?, 1)$; the homotopy relation is an equivalence relation. If $X' \subseteq X$ is a subspace, and if $f|X' = g|X'$, a homotopy is called *relative to this subspace*, iff $F|X' \times t = f|X' = g|X'$, all $t \in I$.

Lemma 107 ([993, Lemma 2, p.130]) *Contiguous simplicial maps which agree on a subcomplex define contiguous maps which are homotopic relative to the space of the subcomplex.*

H.3 Simplicial Coefficient Systems

A simplicial complex K can be viewed as a category whose objects are the simplexes s of K, and whose morphisms are the inclusions $s \subseteq t$ of simplexes. For a commutative ring R, a *coefficient system* of R-modules is a covariant functor $M : K \to {}_R\mathbf{Mod}$ with values in the category ${}_R\mathbf{Mod}$ of R-modules and affine homomorphisms. Let $\Delta_q = \{0, 1, 2, \dots q\}$ be the standard simplex of dimension q. A *singular simplex of dimension* q is a simplicial map $s : \Delta_q \to K$, i.e., a sequence $s_0, s_1, \dots s_q$ of points in K which define a simplex. If we have any set map $f : \Delta_p \to \Delta_q$, we have the singular p-simplex $\overline{f}(s) = s \circ f : \Delta_p \to K$. For a singular simplex s, we denote $M(s) = M(Im(s))$. Clearly $Im(\overline{f}(s)) \subseteq Im(s)$. Therefore we have an affine homomorphism $f_s : M(\overline{f}(s)) \to M(s)$.

Denote by $S_n(K)$ the set of singular simplexes of dimension n in K. Then we have a module $C^n(K; M) = \prod_{s \in S_n(K)} M(s)$, whose elements are called the *singular cochains of dimension* n. For a map $f : \Delta_p \to \Delta_q$, we have an affine map

$$\overline{f} : C^p(K; M) \to C^q(K; M) \tag{H.1}$$

which has $\overline{f}((a_s)_{s \in S_p(K)}) = (b_t)_{t \in S_q(K)}$ and $b_t = f_t(a_{\overline{f}(t)})$. In other words, $C^*(K; M) = (C^n(K; M))_n$ is a *simplicial cochain complex.*

H.3.1 Cohomology

Suppose now that the simplicial cochain complex stems from a system of coefficients with linear maps. Then all the transition maps of equation (H.1) are linear. Consider now the strictly increasing i^{th}-face maps $F^i_n : \Delta_{n-1} \to \Delta_n$ leaving aside index i in Δ_n, i.e., mapping Δ_{n-1} onto the subset $\{0, 1, 2, \dots \hat{i} \dots n\}$. Then we have the coboundary map

$$\begin{aligned} &d_n : C^n(K; M) \to C^{n+1}(K; M), \\ &d_n(a) = \sum_{j=0}^{n+1} (-1)^j \overline{F^j_{n+1}}(a), \end{aligned} \tag{H.2}$$

and $d_{n+1} \circ d_n = 0$. This means that $Im(d_n) \subseteq Ker(d_{n+1})$, and we may consider the *cohomology groups*

$$H^n(K; M) = Ker(d_n)/Im(d_{n-1}) \tag{H.3}$$

for $n \geqslant 0$, with the trivial extension to $C^{-1}(K; M) = 0$.

I

Complements on Calculus

I.1 Abstract on Calculus

I.1.1 Norms and Metrics

Definition 185 *A* pseudo-metric *on a set V is a* (pseudo-distance) *function $d : V \times V \to \mathbb{R}$ such that:*

1. *(Positivity) $0 \leqslant d(x, y)$, and $d(x, x) = 0$ for all $(x, y) \in V \times V$;*
2. *(Symmetry) $d(x, y) = d(y, x)$ for all $(x, y) \in V \times V$;*
3. *(Triangle inequality) $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $(x, y, z) \in V \times V \times V$.*

If conversely $d(x, y) = 0$ implies $x = y$, the pseudo-metric (pseudo-distance function) *is called a* metric (distance function).

Definition 186 *For a pseudo-metric space (X, d), if $0 < r, x \in X$, the* open ball *of radius r around x is $B_r(x) = \{y | d(y, x) < r\}$. The system of open balls*

$$\{B_r(x) | 0 < r, x \in X\}$$

is a base of a topology[1]*, the* (uniform) topology associated with the pseudo-metric d. *Evidently, this topology is Hausdorff iff the pseudo-metric is a metric.*

A map $f : V \to V$ of a pseudo-metric space V is called an isometry *iff $d(f(x), f(y)) = d(x, y)$, for all $(x, y) \in V \times V$.*

Lemma 108 ([163, Lemma 4]) *Given an action $\mu : G \times V \to V$ of a group G on a pseudo-metric space (V, d) by isometries, then*

$$\inf_{g \in G} d(g.x, y) = \inf_{g \in G} d(g.x', y) \quad \text{whenever } G.x = G.x', \tag{I.1}$$

$$\inf_{g \in G} d(g.x, y) = \inf_{g \in G} d(g.x, y') \quad \text{whenever } G.y = G.y'. \tag{I.2}$$

With the above notation, we may define

$$d^*(G.x, G.y) = \inf_{g \in G} d(g.x, y), \tag{I.3}$$

and Lemma 108 guarantees that this is a well-defined function $d^* : G\backslash V \times G\backslash V \to \mathbb{R}$.

[1] In fact, the system $\{B_r = \{(x, y) \in V^2 | d(x, y) < r\} | 0 < r\}$ is a base of a uniformity.

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Definition 187 *If $d(x,y)$ is a pseudo-metric on a set V, and if we have a group action $G \times V \to V$, we say that g acts by isometries, iff each map $g. : V \to V$ is an isometry, i.e., iff $d(g.x, g.y) = d(x,y)$, for all $x, y \in V, g \in G$.*

Lemma 109 ([163, Lemma 5]) *Let d be a pseudo-metric on V, and $\mu : G \times V \to V$ a group action by isometries. Then the function d^* defined in* (I.3) *is a pseudo-distance on the orbit space $G\backslash V$.*

I.1.2 Completeness

A *Cauchy sequence* in a uniform space $(X, \mathcal{U})$ is a sequence $(x_i)_{i=0,1,2,\ldots}$ of elements in X such that for every uniform set $U \in \mathcal{U}$, there is an index t such that $(x_i, x_j) \in U$ for all $i, j > t$. A uniform space is (sequentially) *complete* iff every Cauchy sequence converges.

Lemma 110 *A closed subspace of a complete uniform (in particular: a metric space) space is complete.*

Definition 188 *A* norm *on real vector space X is a function $\| \ \| : X \to \mathbb{R}$ such that for all $(x,y) \in V \times V$:*

1. *(Positivity) $0 \leqslant \|x\|$, and $\|x\| = 0$ iff $x = 0$;*
2. *(Homogeneity) $\|\lambda.x\| = |\lambda|.\|x\|$;*
3. *(Triangle inequality) $\|x + y\| \leqslant \|x\| + \|y\|$.*

Every norm gives rise to an associated metric *$d(x,y) = \|x - y\|$, and therefore to an* associated topology. *A normed vector space with a complete associated (uniform) topology is called a* Banach space.

Example 134 On $\mathbb{R}^n$, we have three well-known norms. If $x = (x_1, \ldots x_n) \in \mathbb{R}^n$, then

1. the absolute or 1-norm is $\|x\|_1 = \sum_i |x_i|$,
2. the Euclidean norm is $\|x\|_2 = \sqrt{\sum_i x_i^2}$,
3. the uniform norm is $\|x\|_\infty = max\{|x_i|| \ i = 1, \ldots n\}$.

For real numbers $a < b$, we have the vector space $C^0[a,b]$ of continuous real-valued functions on the interval $[a,b]$. On $C^0[a,b]$, we have three well-known norms (corresponding to the above three norms). For $f \in C^0[a,b]$, we have:

1. the absolute or 1-norm is $\|f\|_1 = \int_a^b |f|$,
2. the Euclidean norm is $\|f\|_2 = (\int_a^b f^2)^{1/2}$,
3. the uniform norm is $\|f\|_\infty = Max_{[a,b]}|f|$.

Two norms $\| \ \|_1, \| \ \|_2$ on a real vector space X are called equivalent iff there are two positive constants a, b such that $\| \ \|_1 \leqslant a.\| \ \|_2, \| \ \|_2 \leqslant b.\| \ \|_1$. Equivalent norms give rise to the same associated uniformities and topologies, so they have the same Cauchy sequences.

Theorem 83 *Any two norms on a finite-dimensional real vector space are equivalent.*

See [617, Th.3.4.1] for a proof. The theorem implies that every finite-dimensional normed real vector space is Banach, since the standard $\mathbb{R}^n$ is so under the Euclidean norm. We shall therefore mainly work in $\mathbb{R}^n$.

I.1.3 Differentiation

We say that two functions $f, g : U \to \mathbb{R}^m$ that are defined in a neighborhood U of $0 \in \mathbb{R}^n$ define the same *germ* iff they coincide on a common neighborhood of 0. (We are in fact considering the colimit of function spaces on the neighborhood system of 0.) The set of germs in 0 of functions f with $f(0) = 0$ is denoted by F_0. Within this vector space, we have the vector subspace DF_0 of those f with $f(0) = 0$ and $\|f(z)\|/\|z\| \to 0$ if $z \to 0$. We evidently have $Lin_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m) \cap DF_0 = \{0\}$.

Definition 189 *A function $f : U \to \mathbb{R}^m$ which is defined in a neighborhood $U \subseteq \mathbb{R}^n$ of a point x is* differentiable in x *iff there is a linear map $D \in Lin_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m)$ such that $\Delta_x f - D \in DF_0$, where $\Delta_x f(z) = f(x+z) - f(x)$. By the above, D is uniquely determined and is denoted by Df_x. The coefficient of row i and column j of the matrix of Df_x in the canonical basis is denoted by $\partial f_i/\partial x_j$, whereas the matrix is called the* Jacobian of f in x. *A function $f : O \to V$ on an open set $O \subseteq \mathbb{R}^n$ with values in an open set $V \subseteq \mathbb{R}^m$ is differentiable if it is differentiable in each point of its domain O.*

A differentiable function on O defines its *derivative* $Df : O \to Lin_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m) \tilde{\to} \mathbb{R}^{nm}$, which may again be differentiated according to the norm on the space of linear maps. Inductively we define $D^{t+1}f = D(D^t f)$, if it exists. The function f is $\mathcal{C}^r$ iff all derivatives $Df, D^2 f, \ldots D^r f$ exist and are continuous, $\mathcal{C}^0$ denotes just the set of continuous functions. This definition is however not in the right shape for functorial behavior. One therefore adds the linear behavior to the function as follows: Let $TO = O \times \mathbb{R}^n$ be the *tangent bundle* of the open set O. Then we define $Tf : TO \to T\mathbb{R}^m$ by $Tf(x, u) = (f(x), Df_x(u))$. This implies that if $g : U \to \mathbb{R}^l$ is a second differentiable function on an open set $U \subseteq \mathbb{R}^m$ with $f(O) \subset U$, then $g \circ f$ is differentiable and

$$T(g \circ f) = Tg \circ Tf.$$

So we have a functor $T : f \mapsto Tf$ and the natural transformation $pr_1 : T \to Id$ of first projection. More generally, defining $T^{r+1}f = T(T^r f)$, we also have

$$T^r(g \circ f) = T^r g \circ T^r f.$$

Moreover, if we identify $Lin_{\mathbb{R}}(\mathbb{R}^n, Lin_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m))$ with $Bil_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^m))$, etc. for higher multilinear maps, the higher derivatives $D^r f_x$ identify to r-linear maps $(\mathbb{R}^n)^r \to \mathbb{R}^m$.

Proposition 127 *If f is $\mathcal{C}^r$, then $D^r f_x$ is a symmetric r-linear matrix.*

The category of r times differentiable or $\mathcal{C}^r$ functions has the property that the linear parts of the tangent maps compose as normal linear maps do, and this means that the Jacobians of isomorphisms are invertible quadratic matrices.

A curve in $\mathbb{R}^n$ is a $\mathcal{C}^1$-map $y : U \to \mathbb{R}^n$, $U \subset \mathbb{R}$. Its derivative Dy_t in a point $t \in U$ is a linear map $\mathbb{R} \to \mathbb{R}^n$ which identifies to the image of 1 in $Dy_t(1) \in \mathbb{R}^n$, meaning that the derivative can be identified with a continuous map $y' : U \to \mathbb{R}^n : t \mapsto y'(t) = Dy_t(1)$.

I.2 Ordinary Differential Equations (ODEs)

Throughout this section, D denotes an open set in $\mathbb{R}^n$, and $f : D \to \mathbb{R}^n$ denotes a continuous vector field (a function) with components $f_i, i = 1, \ldots n$.

Definition 190 *Let $\zeta \in \mathbb{R}, \eta \in D$, $J(\zeta)$ an open interval containing ζ, and $U(\eta) \subset D$ an open neighborhood of η in D. Denote by $A(f, \zeta, \eta, J(\zeta), U(\eta))$ the set of all C^1-functions $y : J(\zeta) \to U(\eta)$ such that*

$$y' = f \circ y \text{ and } y(\zeta) = \eta. \tag{I.4}$$

Denote by $B(f, \zeta, \eta, J(\zeta), U(\eta))$ the set of all C^1-functions $y : J(\zeta) \to U(\eta)$ such that

$$y = \eta + \int_\zeta^? f \circ y. \tag{I.5}$$

Lemma 111 *With the above definitions, we have*

$$A(f,\zeta,\eta,J(\zeta),U(\eta)) = B(f,\zeta,\eta,,J(\zeta),U(\eta)).$$

The easy proof is left to the reader.

I.2.1 The Fundamental Theorem: Local Case

The following theorem is called the local case of the fundamental theorem of ordinary differential equations.

Theorem 84 *With the preceding notation and definitions, suppose that f is locally Lipschitz, i.e., for every $x \in D$, there is a neighborhood $U(x) \subset D$ and a positive number L such that $x_1, x_2 \in U(x)$ implies $|f(x_1) - f(x_2)| \leqslant L.|x_1 - x_2|$. Then for any "initial condition" $\zeta \in \mathbb{R}, \eta \in D$, there is an open interval $J(\zeta)$ containing ζ, and an open neighborhood $U(\eta)$ of η such that $A(f,\zeta,\eta,J(\zeta),U(\eta))$ is a singleton. The element of A is called the* local solution of the differential equation $y' = f \circ y$ *at* $J(\zeta), U(\eta)$.

The proof uses Lemma 111 and refers to the set B. In fact, it is shown that the operator

$$T_{\zeta,\eta,f}(y) = \eta + \int_{\zeta}^{?} f \circ y$$

is a contraction, and contractions have a unique fixpoint.

Proposition 128 *Let $T : X \to X$ be a contraction on a complete metric space*[2] *(X,d), i.e., there is a constant $0 < c < 1$ such that $d(T(x),T(y)) \leqslant c.d(x,y)$ for all $x, y \in X$. Then, T has a unique fixpoint $z = T(z)$.*

Proof. It suffices to show that the sequence $(x_n = T^n(x))$ is Cauchy. In fact, setting $k = |n - m|$, we have $d(x_n, x_m) = d(T^n(x), T^m(x)) = c^{Min(n,m)}.d(x, T^k)$. But

$$\begin{aligned} d(x,T^k) &\leqslant d(x,T(x)) + d(T(x),T^2(x)) + \ldots d(x^{k-1},T^k(x)) \\ &\leqslant (1 + c + \ldots c^{k-1})d(x,T(x)) \leqslant \frac{1}{1-c}d(x,T(x)). \end{aligned}$$

So this term is limited, while $c^{Min(n,m)}$ tends to zero as n, m tend to infinity, QED.

Corollary 38 *Let X be a complete metric space, and $B = \overline{B_r(x)}$ the closed ball of radius $r > 0$ around x. Let $T : B \to X$ a contraction with $d(T(x),x) \leqslant (1-c)r, 0 < c < 1$. Then T has a unique fixpoint in B.*

Proof. We know from Lemma 110 that B is complete. Further, for $y \in B$, we have $d(T(y),x) \leqslant d(T(y),T(x)) + d(T(x),x) \leqslant c.d(y,x) + (1-c)r \leqslant r$. Therefore, T leaves B invariant and the claim follows from Proposition 128, QED.

Corollary 39 *With the notation of Corollary 38, suppose that $T : B_r(x) \to X$ is a contraction with $d(T(x),x) < (1-c)r, 0 < c < 1$. Then there is a unique fixpoint of T in $B_r(x)$.*

Next, we need some auxiliary results concerning uniform convergence of continuous functions. Let W be a Banach space (in our case $W = \mathbb{R}^n$), A a set, then we set

$$B(A,W) = \{f : A \to W | \|f\|_\infty < \infty\}.$$

Proposition 129 *The set $B(A,W)$ with the usual scalar multiplication and addition of functions is a Banach space.*

[2] See this Appendix, Section I.1.1.

It is clear that $B(A,W)$ is a vector space. Let $(f_n)_n$ be a Cauchy sequence in $B(A,W)$. Since for any $x \in A$, $\|f_n(x) - f_m(x)\| \leqslant \|f_n - f_m\|$, and the right term converges to zero, the left term is also a Cauchy sequence in W and converges to $lim_{n\to\infty} f_n(x) = f(x)$. We first show that $lim_{n\to\infty} f_n = f$. For $0 < \epsilon$, let N be such that $n, m > N$ implies $\|f_m - f_n\| < \epsilon$. Then by definition, for all $x \in A$, $\|f_n(x) - f(x)\| = \|f_n(x) - lim_{m>N} f_m(x)\| = lim_{m>N}\|f_n(x) - f_m(x)\| \leqslant \epsilon$. Therefore $\|f_n - f\| \leqslant \epsilon$, and $f = (f - f_n) + f_n$ is a sum of two elements of $B(A,W)$ and therefore lives in $B(A,W)$, whereas $lim_{n\to\infty} f_n = f$, QED.

Theorem 85 *Let A be a metric space, W a Banach space, and let*

$$BC(A,W) = B(A,W) \cap C^0(A,W)$$

be the set of continuous functions with limited norm. Then $BC(A,W) \subset B(A,W)$ is a closed sub-vector space, and therefore also Banach.

Proof. It is clearly a sub-vector space. Let (f_n) be a Cauchy sequence in $BC(A,W)$. It converges to f in $B(A,W)$. We have to show that it is also continuous. In fact, given $0 < \epsilon$ select n such that $\|f - f_n\| < \epsilon/3$. Let $a \in A$. Take $0 < \delta$ such that $d(x,a) < \delta$ implies $\|f_n(x) - f_n(a)\| < \epsilon/3$. Then $\|f(x) - f(a)\| \leqslant \|f(x) - f_n(x)\| + \|f_n(x) - f_n(a)\| + \|f_n(a) - f(a)\| < \epsilon/3 + \epsilon/3 + \epsilon/3$, QED.

We are now ready for the proof of the local theorem. Recall that we are given a locally Lipschitz vector field function $f : D \to \mathbb{R}^n$. Consider the Banach space $BC = BC(J(\zeta), \mathbb{R}^n)$ for an interval $J(\zeta)$ whose length δ will be determined in the course of the proof. Select $0 < r$ such that (1) the closed ball $B_r(\eta)^- \subset D$, and (2) $f|B_r(\eta)^-$ is Lipschitz with a constant L. Then f is evidently limited on $B_r(\eta)^-$, let m be an upper bound. Let $\bar{\eta} : J(\zeta) \to B_r(\eta)^- : t \mapsto \eta$ be the constant map. Consider the closed ball $B_r(\bar{\eta})^- \subset BC$ around $\bar{\eta}$. For every $g \in B_r(\bar{\eta})^-$, $f \circ g : J(\zeta) \to \mathbb{R}^n$ lives in BC.

We now show that the operator $T(g) = \eta + \int_\zeta^? f \circ g$ defines a contraction

$$T : B_r(\bar{\eta})^- \to BC$$

with contraction constant c such that $d(T(\bar{\eta}), \bar{\eta}) < (1-c)r$. According to Corollary 38, this will imply that T has a unique fixpoint in $B_r(\bar{\eta})^-$ and we are done.

Evidently, $T(g)$ is continuous. Further, for any $x \in J(\zeta)$, we have $|T(g)(x)| \leqslant |\eta| + |\int_\zeta^x f \circ g| \leqslant |\eta| + |x - \zeta|.\|f \circ g\|_\infty$ which evidently is finite.

We are left with the contraction claims. We have

$$\begin{aligned}\|T(\bar{\eta}) - \bar{\eta}\|_\infty &= lub_{J(\zeta)}\|T(\bar{\eta})(t) - \bar{\eta}(t)\| = lub_{J(\zeta)}\|\int_\zeta^t f(\eta)\| \\ &= lub_{J(\zeta)}|t - \zeta|.|f(\eta)| \leqslant \delta.|f(\eta)| \leqslant \delta.m.\end{aligned}$$

For two functions $g_1, g_2 \in B_r(\bar{\eta})^-$, we have

$$\begin{aligned}\|T(g_1) - T(g_2)\| &= lub_{J(\zeta)}|\int_\zeta^t f(g_1) - f(g_2)| \leqslant \delta.\|f \circ g_1 - f \circ g_2\| \\ &= \delta.lub_{J(\zeta)}|f(g_1(s)) - f(g_2(s))| \leqslant \delta.L.lub_{J(\zeta)}|g_1(s) - g_2(s)| \\ &= \delta.L.\|g_1 - g_2\|.\end{aligned}$$

This means that T is a contraction with $c = \delta.L$ if δ is such that $\delta.L < 1$. Further, we need $\delta.m < (1-c)r = (1 - \delta.L)r$, i.e., $\delta < \frac{r}{m + Lr}$ solves the problem, QED.

I.2.2 The Fundamental Theorem: Global Case

The global fundamental theorem deals with maximal integral curves $y : J \to D$ for the differential equation $y' = f \circ y$.

Definition 191 *We say that $u \sim v$ for $u, v \in D$ iff there is a curve $y : J \to D$, defined on an open interval J for the differential equation $y' = f \circ y$, and such that $\{u, v\} \subset y(J)$.*

Lemma 112 *The relation $\sim$ is an equivalence relation. The equivalence class of an element $x \in D$ is denoted by $[x]$.*

It is clearly reflexive and symmetric. It is transitive for the following reason. Let $y_i : J_i \to D, i = 1, 2$ be two integral curves such that $y_1(t_1) = x, y_1(t_2) = y, y_2(t_3) = y, y_2(t_4) = z$. By an evident parameter shift, we may suppose $t_2 = t_3$. We claim that

$$y_1|J_1 \cap J_2 = y_2|J_1 \cap J_2.$$

Suppose that $y_1(t) \neq y_2(t)$ for a $t > t_2$. Let $t_2 \leqslant t_0$ be the infimum of these t. Since our curves are continuous, we have $y_1(t_0) = y_2(t_0)$. But then, according to the local Theorem 84, there is an ϵ-ball $U_\epsilon(t_0)$ around t_0 and a neighborhood $U(y_1(t_0) = y_2(t_0))$ such that there is a unique integral curve $y : U_\epsilon(t_0) \to U(y_1(t_0))$. But we may suppose WLOG that ϵ is so small that both $y_1|U_\epsilon(t_0), y_2|U_\epsilon(t_0)$ have their codomains in $U(y_1(t_0))$. Evidently, these solutions must then coincide with the unique solution on the open interval $U_\epsilon(t_0)$, but this contradicts the choice of t_0. A symmetric argument holds for the supremum $s_0 \leqslant t_2$ of those arguments with $y_1(t) \neq y_2(t)$. Therefore, $y_1|J_1 \cap J_2 = y_2|J_1 \cap J_2$, and we may extend the integral curves y_1, y_2 to the domain $J_1 \cup J_2$, whence the transitivity of the $\sim$-relation, QED.

Theorem 86 *Let $x \in D$. Then there is a unique integral curve $y : J \to D$ with $y(0) = x, y' = f \circ y$, and such that J contains all domains of any integral curve $z, z(0) = x, z' = f \circ z$. We have $y(J) = [x]$ and write $\int_x f$ for this curve; it is called the* global solution through x.

Proof. Let $\Gamma = \{\Gamma_{y_i} \subset \mathbb{R} \times D | \Gamma_{y_i} =$ graph of solution y_i of $y_i' = y_i \circ f, x = y_i(0)\}$. Since two solutions coincide on the intersection of their domains, the union $\bigcup \Gamma$ is functional, and the union of the domains is an open interval J. Further, the function y of this graph is a solution of the differential equation $y' = f \circ y$ which reaches all elements equivalent to x, QED.

Corollary 40 *Let $x_1 \sim x_2$ and $\int_{x_1} f(t_2) = x_2$. Then $\int_{x_2} f = \int_{x_1} f \circ e^{t_2}$ and $J_2 = e^{-t_2} J_1$.*

Definition 192 *The quotient $D/\sim = \{Im(\int_x f) | x \in D\}$ is called* the phase portrait of the vector field f *and denoted by D/f. An integral curve which is not an injective function of its parameter is called* a cycle of the field f.

Proposition 130 *Let $\int_w f$ be a cycle with $\int_w f(t_1) = \int_w f(t_1 + T)$. Then the cycle's domain is $\mathbb{R}$ and $\int_w f$ is T-periodic.*

Proof. Let $y : J \to D$ be the cycle $\int_w f$ with $y(t_1) = z$. Consider the function $\hat{y} = y \circ e^T : J - T \to D$. Evidently, $\hat{y}(t_1) = y(t_1)$, and $\hat{y}$ also solves the differential equation since $\hat{y}'(t) = y'(t + T) = f \circ y(t + T) = f \circ \hat{y}(t)$. So, since $\hat{y}$ has a common value with y at t_1, by maximality of y, we have $\hat{y} = y|J - T$, and $J - T \subset J$, whence $J =]-\infty, b[$. Symmetrically, exchanging t_1 with $t_1 + T$, and T with $-T$, we obtain $J =]a, \infty[$, i.e., $J = \mathbb{R}$. Now, for any $t \in \mathbb{R}$, with $\hat{y} = y \circ e^T$, uniqueness guarantees $\hat{y} = y$, whence the periodicity of y, QED.

Proposition 131 *Suppose that D^- is compact (e.g.: D is bounded), and that f is locally Lipschitz on D^-. If the domain $J =]a, b[$ of a maximal curve $\int_x f$ has finite upper bound b, then $t \to b$ implies $\int_x f(t) \to \partial D$*

Sketch of proof: Write $y = \int_x f$, and suppose that the closure of $y(J)$ were in D. Then, since $y(J)^-$ is compact, there is a convergent sequence $t_n \to b$ with a convergent image sequence $y(t_n) \to q, q \in D$. It can be shown that q is uniquely determined, i.e., another such sequence yields the same limit. We then set $y(b) = lim_{t \to b} y(t) = q$, and y may be extended to a local solution containing b, a contradiction, QED.

I.2.3 Flows and Differential Equations

On an open set $O \subseteq \mathbb{R}^n$, a vector field (a $\mathcal{C}^1$-map) $f : O \to \mathbb{R}^n$ can also be viewed by its graph as a section $O \to TO : x \mapsto F(x) = (x, f(x))$. If $x \in O$, an *integral curve of F in x* is a curve $y : U(0) \to O$ defined on an open neighborhood $U(0)$ of 0 such that $y(0) = x$ and $y' = f \circ y$. By the main Theorem 86 of ODEs, there is a unique maximal integral curve $\int_x f$ for every point $x \in O$.

For a vector field F on $O \subseteq \mathbb{R}^n$, a *flow box* is a triple (U, a, W) where $U \subseteq O$ is open, a is a positive real number of ∞, and $W : U \times]-a, a[\to O$ is $\mathcal{C}^1$ such that for all $x \in U$, $W_x :]-a, a[\to O : t \mapsto W(x,t)$ is an integral curve of F at x. Two flow boxes $(U, a, W), (U', a', W')$ always coincide in their maps W, W' on the intersection $(U \cap U') \times (]-a, a[\cap]-a', a'[)$ of their domains. For each point $x \in O$, there is a flow box (U, a, W) with $x \in U$. Let $\mathcal{D}_F = \{(x,t) \in \mathbb{R} \times O |$ there is an integral curve $\int_x$ whose domain contains $t\}$. Then (1) $\mathcal{D}_F$ is open in $\mathbb{R} \times \mathbb{R}^n$; (2) there is a unique map $W_F : \mathcal{D}_F \to O$ such that $t \mapsto W_F(t,x)$ is an integral curve at x for all $x \in O$.

I.2.4 Vector Fields and Derivations

For a $\mathcal{C}^1$-function $f : O \to \mathbb{R}$, we have the derivative $Tf : TO \to T\mathbb{R}$, whose second component evaluates to linear forms on $\mathbb{R}^n$. This map $df = pr_2 \circ Tf = Df$ is called the *differential of f*. If $F : O \to TO$ is a vector field, the composition $L_F f = df \circ F : O \to \mathbb{R}$ is called the *Lie derivative of f with respect to F*. If we denote by $grad(f)$ the differential of f as a tangent vector $(\partial_{x_1} f, \dots \partial_{x_n} f)$ (the old-fashioned gradient of f), the Lie derivative is just the scalar product of $grad(f)$ with the vector field. If $\mathcal{F}(O)$ denotes the real algebra[3] of $\mathcal{C}^1$-function on O, the map $L_F : \mathcal{F}(O) \to \mathcal{F}(O)$ is a *derivation* in the sense that:

(i) L_F is linear;
(ii) for $f, g \in \mathcal{F}(O)$, we have $L_F(f.g) = f.L_F(g) + L_F(f).g$;
(iii) If $c \in \mathcal{F}(O)$ is constant, then $L_F c = 0$.

Therefore, we also have $d(f.g) = df.g + f.dg$ and $dc = 0$ for a constant c. Denote by $VF(O)$ the vector space of all $\mathcal{C}^\infty$-vector fields on O. Then:

Theorem 87 *The Lie map*

$$L_? : VF(O) \to Der(\mathcal{F}(O)) : F \mapsto L_F$$

is an isomorphism of vector spaces.

See [2, Th.8.10] for a proof.

In particular, the Lie bracket $[L_F, L_G] = L_F \circ L_G - L_G \circ L_F$ which is a derivation, must be the Lie derivative of a unique vector field which is denoted by $[F, G]$, the *Lie bracket of the vector fields F and G*. The Lie bracket makes the vector space $VF(O)$ into a real Lie algebra, see Section E.4.4.

I.3 Partial Differential Equations

For this section, refer to [509].

We only need a short review of quasi-linear first order partial differential equations (PDE). Recall that a PDE is an equation of type $E(x_1, x_2, \dots u, u_{x_1}, u_{x_2}, \dots u_{x_1 x_1}, u_{x_1 x_2}, \dots) = 0$ where u is a function of the n real variables $x_1, x_2, \dots x_n$, with its partial derivatives $u_{x_1}, \dots$, the higher partial derivatives $u_{x_1 x_1} \dots$ etc. A solution is meant to be such a function u which is defined in an open set O of $\mathbb{R}^n$. Its *order* m is the highest number of iterated partial derivatives, whereas E is called *quasi-linear* iff it is an affine function of the derivatives of u of highest order m, with coefficients that are functions of the variables $x_1, x_2, \dots u, u_{x_1} \dots$ until derivatives of order $m-1$.

A first-order quasi-linear PDE has the shape

[3] Multiplication goes pointwise.

$$\sum_i a_i(x_1, x_2, \dots u) u_{x_i} = c(x_1, x_2, \dots u)$$

and can be solved by a system of ODEs, this is the *method of characteristics*.

We illustrate the method for two variables, i.e., for the equation

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u). \tag{I.6}$$

The solution $u(x, y)$ is represented as a surface $z = u(x, y)$ in $\mathbb{R}^3$. Such a surface is called an *integral surface* of the equation (I.6). We have a vector field

$$F(x, y, z) = (a(x, y, z), b(x, y, z), c(x, y, z))$$

on the common domain U of the three functions a, b, c. The tangent space of an integral curve at $x, y, u(x, y)$ is spanned by the vectors $X = (1, 0, u_x)$ and $Y = (0, 1, u_y)$. Their vector product $Y \wedge X = (u_x, u_y, -1)$ is the normal vector to the integral surface. Therefore equation (I.6) just means that the scalar product $(F(x, y, z), Y \wedge X)$ vanishes identically, i.e., the vector field F is tangent to the integral surface. Clearly, only the direction of the vectors of the vector field F, the *characteristic directions* matter for the equation (I.6). It is easily seen that an integral curve of F, if it crosses a point of an integral surface, is entirely contained in this surface. Therefore an *integral surface is the union of integral curves of the directional vector field* F. An integral surface can be constructed by finding a curve Γ which lies in an integral surface, and which is never parallel to an integral curve of F. This is the *Cauchy problem for the equation* (I.6). Then, the parameter of Γ and the curve parameter of a flow box (see Section I.2.3) around Γ describe the integral surface. Technically, the existence condition for $\Gamma(t) = (\Gamma_x(t), \Gamma_y(t), \Gamma_z(t))$ to generate a surface is that the projection $\Gamma_{xy}(t) = (\Gamma_x(t), \Gamma_y(t))$ is never parallel to the projection F_{xy} of the directional field on the xy plane. The existence of a curve Γ is again guaranteed by the main theorem of ODEs, and we are done.

J

More Complements on Mathematics

Summary. This appendix is not self-contained, it completes the mathematical Appendix Part XXI and is built upon those topics.

– Σ –

J.1 Directed Graphs

Directed graphs, also called *digraphs* or, in Gabriel's language *quivers*, define a basic category in mathematics and computer science. Their category is also one of the most accessible topoi (see Appendix Section G.3 for the concept of a topos). The sketchy Appendix Section C.2.2 about graphs and quivers can be skipped when reading the following section.

J.1.1 The Category of Directed Graphs (Digraphs)

The category **Digraph** of directed graphs, short: digraphs. This is a basic category for algebra as well as for topology. In the naive setup, its objects are functions $\Gamma : A \to V^2$ from a set $A = A_\Gamma$ of *arrows* to the Cartesian square $V^2 = V \times V$ of the set $V = V_\Gamma$ of *vertices*. The first projection $t = pr_1 \circ \Gamma$ is called the *tail* function, the second $t = pr_2 \circ \Gamma$ is called the *head* function of the digraph. For an arrow a, the vertices $t(a)$, $h(a)$ are called its head and tail, respectively, and denoted by $t(a) \xrightarrow{a} h(a)$. A morphism $f : \Gamma \to \Delta$ of digraphs is a couple $f = (u, v)$ of functions $u : A_\Gamma \to A_\Delta, v : V_\Gamma \to V_\Delta$ such that $v^2 \circ \Gamma = \Delta \circ u$.

Example 135 The *initial object* in the category **Digraph** is the empty digraph $\varnothing$, whose arrow and vertex sets are both empty. The *terminal object* in **Digraph** is the digraph $1 = t \circlearrowright T$ with one vertex t and a single loop T on t. A third important elementary digraph is the *arrow digraph* $\uparrow = \bullet \to \bullet$, having two vertices and one connecting arrow. A generalization of $\uparrow$ is the n-fold concatenation $\uparrow^n$ of $\uparrow$, consisting of $n + 1$ vertices, and having one arrow from vertex i to vertex $i + 1$ for all $i = 0, 1, 2, \ldots n - 1$. One also extends this power to $n = 0$, meaning that $\uparrow^0$ is the digraph with one single vertex and no arrow.

Example 136 Denote the switch bijection on V^2 which maps (x, y) to (y, x) by ?*. Then for every digraph $\Gamma : A \to V^2$, we have its *dual digraph* Γ^* that is defined by $\Gamma^* = ?^* \circ \Gamma$. Evidently, $\Gamma^{**} = \Gamma$. The dual digraph construction defines an functor automorphism on **Digraph**.

Example 137 In this book, we often need a special subcategory of digraphs, the *spatial digraphs*. Such a digraph is associated with a topological space X (see Appendix Section H.1) and is denoted by $\overrightarrow{X}$. By definition, the arrow set is $A_{\overrightarrow{X}} = I@_{\mathbf{Top}}X$, the set of continuous curves $c : I = [0, 1] \to X$ in X, while the vertex set is $V_{\overrightarrow{X}} = X$, with $h(c) = c(1)$, and $t(c) = c(0)$. A spatial morphism is a digraph morphism

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$\overrightarrow{f} : \overrightarrow{X} \to \overrightarrow{Y}$ canonically induced by a continuous map $f : X \to Y$. The subcategory of spatial digraphs and morphisms is denoted by *SpaceDigraph*.

A spatial digraph is more than a digraph. It is also a topological digraph in the following sense. The set $A_{\overrightarrow{X}} = I@X$ of arrows of $\overrightarrow{X}$ is a topological space by the compact-open topology (see Appendix Section J.4.1.2), and the head and tail maps $h, t : I@X \to X$ are continuous. Moreover, for a continuous map $f : X \to Y$, the arrow map $I@f : I@X \to I@Y$ is continuous.

Every digraph Γ gives rise to a (small) category, the *path category* $Path(\Gamma)$. It is defined as follows.

Definition 193 *A path of length $n \geqslant 0$ in a digraph Γ is a morphism $p :\uparrow^n \to \Gamma$, we the write $l(p) = n$ for the* length *of p. A path of length zero is also called a* lazy path*. It is given by the single vertex $p(0)$ in Γ. The first vertex of a path p is denoted by $d(p)$ and called the path's* domain*, the last vertex of p is denoted by $c(p)$ and called the path's* codomain*. A path p with $d(p) = c(p)$ is called a* cycle*, and if $l(p) = 1$, the cycle is called a* loop*. A vertex in Γ that is not the tail of an arrow is called a* leaf of Γ*. A vertex v of Γ that has a path p for every vertex w such that $d(p) = v, c(p) = w$ is called a* root or source *of Γ, and if v is a root of the dual Γ^*, it is called a* co-root or sink *of Γ. A digraph with a (necessarily unique) root and without directed cycles is called a* directed tree*.*

The category $Path(\Gamma)$ has the paths $p :\uparrow^n \to \Gamma$ as their morphisms. If $d(q) = c(p)$, then we may concatenate q after p in the evident way, yielding a path $q \circ p$ of length $l(q \circ p) = l(q) + l(p)$, the *composition of p with q*. This defines the category $Path(\Gamma)$, the lazy paths being its objects.

J.1.1.1 Unordered Graphs

Unordered graphs will play a minor role in this book, so we keep the information minimal here. Whereas a digraph needs the cartesian product $V \times V = V^2$ of its vertex set V, an undirected graph, or simple *graph*, needs the *edge set* ${}^2V = \{a \subset V | 1 \leqslant card(a) \leqslant 2\}$. These sets a represent intuitively edges between vertices, and a loop in case $card(a) = 1$. There is an evident surjection $|V| : V^2 \to {}^2V : (x, y) \mapsto \{x, y\}$, which has a number of sections that we denote somewhat ambigusouly by $\overrightarrow{V} : {}^2V \to V^2$, i.e., $|V| \circ \overrightarrow{V} = Id\,{}_{^2V}$.

Definition 194 *A* graph *(or an* unordered graph*) is map $\Gamma : A \to {}^2V$. The elements of A are called* edges*, the elements of V are called* vertices *of Γ. For $a \in A$ and $\Gamma(a) = \{x, y\}$, we also write $x \overset{a}{\text{——}} y$, which is the same as $y \overset{a}{\text{——}} x$.*

Much as digraphs, graphs also have morphisms as follows.

Definition 195 *If $\Gamma : A \to {}^2V$ and $\Delta : B \to {}^2W$ are two graphs, a morphism $f : \Gamma \to \Delta$ is a couple $f = (u, v)$ of maps, $u : A \to B, v : V \to W$ such that ${}^2v \circ \Gamma = \Delta \circ u$.*

Graphs and their morphisms the clearly define the category *Graph* of graphs. Moreover, the surjections $|V|$ define a functor $|?| :$ **Digraph** $\to$ *Graph*, and the graph $|?|(\Gamma)$ is called the *associated graph of Γ* and denoted by $|\Gamma|$. Conversely, given a graph Δ, a digraph Γ such that its associated graph $|\Gamma|$ is Δ is called *associated digraph of Δ*, even though this one is not uniquely determined.

Definition 196 *Similar to the chain digraphs $\uparrow^n$, we have chain graphs, which are just the associated graphs $|\uparrow^n|$, we denote them by $|^n$. Corresponding to directed paths we have undirected paths in a graph Δ, called* walks*, namely the morphisms $w : |^n \to \Delta$. A (undirected) cycle in graph Δ is a walk that starts where it ends, and a lazy walk is a cycle of length 0. An* undirected tree *is a graph that has no undirected cycles of positive length and a (automatically uniquely determined) root. Undirected trees are associated with directed trees, bot not vice versa. A* spanning tree *in a graph Δ is a subgraph that is a tree and contains all vertices of Δ.*

Here is a useful lemma:

Lemma 113 *Every connected graph (any two vertices admit a walk that connects them) admits a spanning tree.*

J.1.2 Two Standard Constructions in Graph Theory

The following constructions of digraphs are crucial in graph theory, and will be used in section J.1.3.

Proposition 132 *For any three digraphs Γ, Δ, Σ and a couple of morphisms $\Gamma \to \Delta, \Delta \to \Sigma$, there is a fiber product $\Gamma \times_\Sigma \Delta$. For any three digraphs Γ, Δ, Σ and a couple of morphisms $\Sigma \to \Gamma, \Sigma \to \Delta$, there is a fiber sum $\Gamma \sqcup_\Sigma \Delta$.*

These constructions are meant in the sense of category theory, sharing the universal properties. Here are the two constructions for the simpler case where $\Sigma = \varnothing$, the general case works similerly: If $\Gamma : A \to V^2, \Delta : B \to W^2$, then we have the cartesian product map $A \times B \to V^2 \times W^2$, but we have a canonical bijection $V^2 \times W^2 \xrightarrow{\sim} (V \times W)^2$, which yields a digraph $\Gamma \times \Delta : A \times B \to (V \times W)^2$, this is the cartesian product digraph. On the other hand, we have the coproduct map $A \sqcup B \to V^2 \sqcup W^2$, and there is an evident injection $V^2 \sqcup W^2 \to (V \sqcup W)^2$, whence the coproduct digraph $\Gamma \sqcup \Delta : A \sqcup B \to (V \sqcup W)^2$.

As is well known from category theory (see Appendix Section G.2.2), if we have fiber products and fiber sums, and also initial and final objects, then all finite limits and colimits also exist, whence this theorem:

Theorem 88 *The category* **Digraph** *is finitely complete and co-complete.*

Remark 34 A special case is this fact: Any digraph Δ is the colimit of the following diagram $\mathcal{D}$ of digraphs: We take one arrow digraph $\uparrow_a = \uparrow$ for each arrow $a \in A_\Delta$ and one bullet digraph $\bullet_x = \bullet$ for each vertex $x \in V_\Delta$. We take as morphisms the tail or head injections $\bullet_x \to \uparrow_a$ whenever $x = t(a)$ or $x = h(a)$. Then evidently, $\Delta \xrightarrow{\sim} \mathrm{colim}\mathcal{D}$.

J.1.3 The Topos of Digraphs

In this section we want to prove that **Digraph** is a topos. Recall from Definition 170 the concept of a(n elementary) topos. Let us recapitulate a short definition (Appendix Section G.3):

Definition 197 *A topos is a category $\mathcal{C}$ that*

(*i*) **C** *is cartesian closed,*
(*ii*) **C** *has a subobject classifier $1 \rightarrowtail \Omega$.*

Let us recall these two properties from Appendix Section G.3.

Definition 198 *A category $\mathcal{C}$ is* cartesian closed *if it has finite cartesian products (which is equivalent to having binary products and a terminal object), and if every functor $A \times ? : X \mapsto A \times X$ has a right adjoint, denoted by $?^A$, i.e., $(A \times X) @_{\mathcal{C}} Y \xrightarrow{\sim} X @_{\mathcal{C}} Y^A$ as a bifunctor in X and Y.*

Definition 199 *Given a category $\mathcal{C}$ which is finitely complete, with the terminal object 1, a monomorphism $true : 1 \rightarrowtail \Omega$ in $\mathcal{C}$ is called a* subobject classifier *iff given any monomorphism $\sigma : S \rightarrowtail X$ in $\mathcal{C}$, there is a unique morphism $\chi_\sigma : X \to \Omega$ such that the diagram*

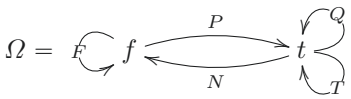
$$\begin{array}{ccc} S & \xrightarrow{\sigma} & X \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \chi_\sigma} \\ 1 & \xrightarrow{true} & \Omega \end{array} \tag{J.1}$$

is a fiber product.

There is an easy proof of the fact that **Digraph** is a topos. It resides on a more creative redefinition of **Digraph** as follows. We may view it as the category of presheaves[1] $ht^{@}$ over the small category $ht = \bullet \underset{t}{\overset{h}{\rightrightarrows}} \bullet$ with two objects and just two parallel morphisms h, t between these points. Then our claim follows immediately[2] from a general proposition:

Proposition 133 *For a category $\mathcal{C}$ the presheaf category $\mathcal{C}^{@}$ is a topos.*

The description of the subobject classifier in **Digraph** is as follows. The final object $1 = t \circlearrowright$ is embedded by the true morphism $T : 1 \to \Omega$ into the subobject classifier



Every evaluation $\Gamma @ \Omega$ describes the set of subdigraphs of Γ, together with its canonical Heyting logic (see Appendix Section G.5.1, Theorem 72).

J.2 Galois Theory

This section is not an introduction to classical Galois theory, but rather focuses on the relationship of Galois theory to the classical problems of cube doubling, angle trisection and π construction using compass and straightedge. For a reference to Galois theory, see [1079].

The main theorem of Galois theory deals with extensions of commutative fields an their relations to corresponding groups of field automorphisms. We only deal with commutative fields here. A field extension is symbolized by L/K, this means that we have an inclusion of two fields $K \subset L$. An extension is called algebraic if every element $x \in L$ is algebraic over K, i.e., if it is a zero, $f(x) = 0$, of a polynomial $f(X) \in K[X]$. A non-algebraic element of an extension is called *transcendental* (over K). A finite extension is one where the dimension $[L : K] = dim_K(L) < \infty$. Such an extension is automatically algebraic. A separable extension is an extension L/K such that every element $x \in L$ is a zero of an irreducible polynomial that has no multiple zeros.

Definition 200 *An separable finite field extension L/K is called* Galois *or* normal *iff it verifies one of the following equivalent conditions:*

1. *Every irreducible polynomial $f \in K[X]$ that has a zero in L factorizes in linear factors.*
2. *The field L is generated from K by adjunction of all zeros of a polynomial $f \in K[X]$.*

A Galois field extension L/K always has a *primitive element* ξ, i.e., it is the algebraic extension generated by adjunction of a single element ξ to K. $L = K(\xi)$. The *Galois group* of a Galois extension L/K is the group $Gal(L/K)$ of all field automorphisms $f : L \xrightarrow{\sim} L$ that leave K pointwise fixed, also called *relative automorphisms*. We then have the fact that if $L = K(\xi)$ with ξ being the zero of an irreducible polynomial $g(X)$ of degree n, then we have

$$|Gal(L/K)| = [L : K] = n.$$

This is the main theorem of Galois theory:

[1] For any category $\mathcal{C}$, $\mathcal{C}^{@}$ denotes the category of presheaves, i.e., contravariant set-valued functors, on $\mathcal{C}$. Our notation stems from the Yoneda embedding $y : \mathcal{C} \to \mathcal{C}^{@}$, which associates with an object X in $\mathcal{C}$ the presheaf $@X$ which yields the set $@X(Y) = Y @ X$ of morphisms $f : Y \to X$ in $\mathcal{C}$.

[2] Observe that $ht^{op} \xrightarrow{\sim} ht$, so it is also legitimate to view $ht^{@}$ as being built from the covariant functors on ht.

Theorem 89 *Let $Ex(L/K)$ be the set of intermediate field extensions M, $K \subset M \subset L$. Let $SubGal(L/K)$ be the set of all subgroups H of $Gal(L/K)$. Then we have a bijection*

$$Ex(L/K) \tilde{\to} SubGal(L/K)$$

defined by $K \subset M \subset L$ being sent to the group of relative automorphisms of the extension L/M, while the inverse bijection sends a subgroup $H \in SubGal(L/K)$ to the field of those elements of L that remain fixed under H. We have $|H| = [M : K]$.

The problem of constructing the double of a cube's volume or the trisection of an angle can be reduced to a problem of Galois theory as follows: Suppose the geometric situation of the plane can be described in a coordinate system by a set of numbers that represent points, straight lines (defined by two points) or circles (defined by their center and the diameter), call the field extension $K/\mathbb{Q}$ defined by these numbers K. Then a new geometric object x can be constructed using compass and straightedge if its coordinate numbers can be calculated using rational operations (sum, difference, product, quotient) and square roots since all these operations can be done using compass and straightedge. These operations are not necessarily real ones, but working in $\mathbb{C}$ does not make the problem more difficult since rational operations and square roots on complex numbers can be managed using compass and straightedge, this is an easy exercise. The converse is also true: If we can construct a number using compass and straightedge, we only perform rational operations or calculate square roots. Think of intersecting two curves, a curve and a circle, or two circles.

Theorem 90 *Given the basic data in the field extension $K/\mathbb{Q}$, in order to be constructible using compass and straightedge, a number x must be contained in a Galois field extension L/K with $[L : K] = 2^m$. Conversely, if x is contained in such an extension, it can be constructed using compass and straightedge.*

The second statement follows from the fact that if $[L : K] = 2^m$, then $Gal(L : K)$ is a solvable group which means that it has a composition series $1 \subset H_1 \subset H_2 \subset \ldots Gal(L : K)$ of subgroups H_i that are normal pairs $H_i \lhd H_{i+1}$ with H_{i+1}/H_i being abelian of order a power of 2, which (using the main theorem on finitely generate abelian groups [714, Appendix C.3.4.2]) implies $H_{i+1}/H_i \tilde{\to} \mathbb{Z}_2$, i.e., using the main theorem of Galois theory, the corresponding extensions are quadratic and can be constructed using rational operations and square roots.

But our problems of doubling the volume or angle trisection lead to cubic irreducible polynomials. Their solutions must generate field extensions of K that have degree 3, and such extensions cannot be contained in extensions of degree 2^m, therefore these problems cannot be solved using compass and straightedge. The construction of π is even simpler to deal with if one accepts that π is not even algebraic over K, it is a transcendental number (the proof of this fact is quite hard and not a topic of pure algebra), and therefore cannot be contained in any algebraic extension of K.

J.3 Splines

Splines are functions that are built from a special type of functions (polynomials, exponential functions, etc.) that are defined on simple types of subspaces of their domains, whence the name "spline": these special functions are "splined" together.

J.3.1 Some Simplex Constructions for Splines

Recall from Appendix Section H.2 that the standard (closed) simplex Δ_d of dimension d (also called d-simplex) is the set of points $(\xi_0, \xi_1, \ldots \xi_d) \in \mathbb{R}^{d+1}$ such that $\sum_i \xi_i = 1$ and $0 \leqslant \xi_i \leqslant 1$ for all $i = 0, 1, \ldots d$.

Definition 201 *Given natural numbers n, d and a sequence $(P_i)_{i=0,\ldots d}$ of points $P_i \in \mathbb{R}^n$, the* affine simplex *$\Delta(P_0, \ldots P_d)$ is the unique map*

$$\Delta(P_0, \dots P_d) : \Delta_d \to \mathbb{R}^n$$

with $\Delta(P_0, \dots P_d)(e_i) = P_i$ *for all standard basis vectors* $e_i = (0, \dots 0, 1, 0, \dots 0)$*, which is induced by an affine map* $\mathbb{R}^{d+1} \to \mathbb{R}^n$*. We denote the image of this affine simplex by* $\Delta[P_0, \dots P_d]$*.*

The image is also described by this formula:

$$\Delta[P_0, \dots P_d] = \{\sum_{i=0}^{d} \lambda_i P_i | (\lambda.) \in \Delta_d\}.$$

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is an affine map, then we have (with the above notations)

$$f \circ \Delta(P_0, \dots P_d) = \Delta(f(P_0), \dots f(P_d)).$$

Definition 202 *A subset* $X \subset \mathbb{R}^n$ *is called* convex *iff for all* $x, y \in X$ *the closed line* $\Delta[x, y]$ *is a subset of* X*. For any subset* D *of* $\mathbb{R}^n$*, its* convex hull $Conv(D)$ *is the intersection of all convex sets in* $\mathbb{R}^n$ *containing* D*.* $Conv(D)$ *is in fact the minimal convex set containing* D*.*

In particular,

Proposition 134 *The convex hull of a finite sequence* $(P_i)_{i=0,\dots d}$ *of points* $P_i \in \mathbb{R}^n$ *is the image set* $\Delta[P_0, \dots P_d]$*. More generally, the convex hull of any set* $X \subset \mathbb{R}^n$ *is the union of all affine images* $\Delta[P_0, \dots P_d]$ *of finite sequences* $(P_i)_{i=0,\dots d}$ *of points in* D*.*

Proposition 135 *If* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is an affine map and* $D \subset \mathbb{R}^n$*, then*

$$f(Conv(D) = Conv(f(D)).$$

J.3.2 Definition of General Splines

Given an n-dimensional cube $K(a., b.) = K(a_1, \dots a_n; b_1, \dots b_n)$ for n intervals $a_i \leqslant x_i \leqslant b_i$, denote by $Part(a., b.)$ the set of *partitions of* $K(a., b.)$. A partition is defined as follows: First, a partition of a one-dimensional cube, i.e., an interval $[a, b]$, is a finite subset $P \subset [a, b]$ that contains a and b. This means that we have an increasing set of points $a < x_1 < \dots x_k < b$, the elements of P. Call $vol(x_i, x_{i+1}) = x_{i+1} - x_i$ the *volume* of this interval. An n-dimensional partition is a cartesian product $P = P_1 \times \dots P_n$ of partitions P_i of $[a_i, b_i]$. This means that the elements P_j of P define cartesian products of one-dimensional intervals, closed cubes c, and the volume $vol(c)$ of such a cube is the product of its one-dimensional interval volumes. Denote by $I)(P)$ the set of these cubes of P. Figure J.1 shows a three-dimensional partition.

Let partition P, Q be two partitions in $Part(a., b.)$. We say that Q *is a refinement of* P and write $P \to Q$ if every defining factor P_i of P is a subset of the defining factor Q_i of Q.

The definition of types of functions for splines is less precise, but here is what they mean: If $K \subset \mathbb{R}^n$ is an n-dimensional cube, we consider a set $\mathcal{T}_n(K)$ of functions $f : K \to \mathbb{R}^m$, usually a real vector space (with the point-wise addition and scalar multiplication). Typical examples are polynomial functions, differentiable functions, polynomial functions of limited degree, etc. One prominent example are the functions that are polynomial functions in every coordinate of $\mathbb{R}^m$.

Definition 203 *Given natural numbers* n, m*, a partition* $P \in Part(a., b.)$*, and a type* $\mathcal{T}$ *of functions, a* m-dimensional spline of type $\mathcal{T}$ *is a function* $f : K(a., b.) \to \mathbb{R}^m$ *such that everyone of its restrictions to a cube* $K \in I(P)$ *is of type* $\mathcal{T}$*. If we require an additional condition* C *of* f *about its behavior on the partition's cubes, we say that* f *is a spline of type* $\mathcal{T}$ with condition C*. In particular, if* $n = 1$ *we speak of* spline curves*, for* $n = 2$ *we speak of* spline surfaces*.*

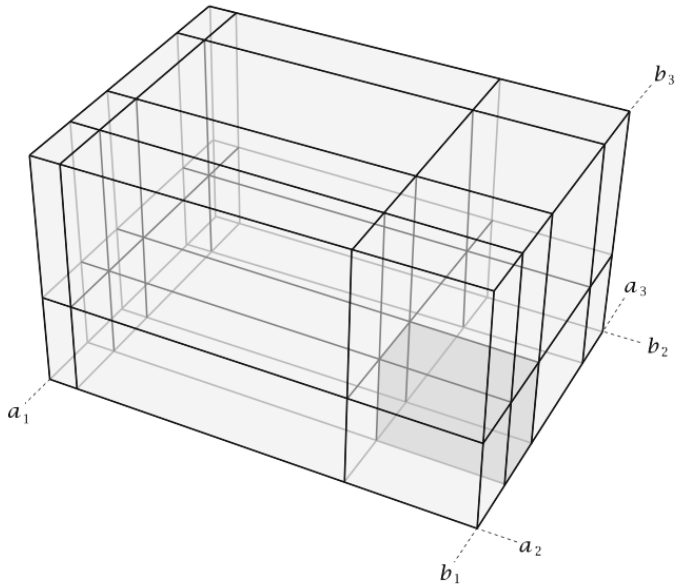


Fig. J.1. A three-dimensional partition.

Let us discuss the popular role of cubic splines, i.e., splines that relate to the function type $\mathcal{T} = Pol_m^3$, the polynomial functions of at most degree 3 in every coordinate of the target space $\mathbb{R}^m$. Let us first consider such a function $f : [0,1] \to \mathbb{R}$ with these conditions:

$$f(0) = y_0, f(1) = y_1, f'(0) = t_0, f'(1) = t_1.$$

If $f(X) = a + bX + cX^2 + dX^3$, then these conditions mean that

$$a = y_0, b = t_0, c = 3(y_1 - y_0) - 2t_0 - t_1, d = 2(y_0 - y_1) + t_0 + t_1.$$

This means that such polynomials allow for a unique solution for values of the functions and their first derivatives on the boundary points of intervals. This implies the general result:

Theorem 91 *For every one-dimensional partition $P = (a = x_0 < x_1 < \ldots x_k = b)$ and two sequences $(y_0, y_1, \ldots y_k), (t_0, t_1, \ldots t_k)$ of vectors in $\mathbb{R}^m$, with the intervals $K_i = [x_i, x_{i+1}], i = 0, \ldots k-1$, there is a unique spline function $f : [a,b] \to \mathbb{R}^m$ of type Pol_m^3 with the condition $f(x_i) = y_i, i = 0, \ldots k$ and such that $Df_{i-1}(x_i) = Df_i(x_i) = t_i, i = 1, \ldots k-1$, and $Df_0(x_0) = t_0, Df_{k-1}(x_k) = t_k$, where $f_j = f|_{K_j}$.*

J.4 Topology and Topological Categories

J.4.1 Topology

J.4.1.1 Generators for Topologies

(Recalling some basic concepts from Appendix Secction H.1) For any non-empty family $(T_i)_i$ of topologies $T_i \subset 2^X$ of a set X, the intersection $\bigcap_i T_i$ is again a topology. If $T \subset 2^X$ is any set of subsets of X, the family of topologies T_i such that $T \subset T_i$ is not empty (the discrete topology is a member), and this family's intersection is called the *topology generated by* T and denoted by $\langle T \rangle$. The elements of T are also called a *subbase* of $\langle T \rangle$. The open sets of $\langle T \rangle$ are the unions of finite intersections of members of T. A set B of open sets of a topology is called a *base* of this topology iff every open set is the union of a family of open sets

from B. For example, the set of finite intersections of members of T in the above example are a base of $\langle T \rangle$. For a point $x \in X$ of a topological space X, a *neighborhood base* is a set of neighborhoods such that every neighborhood of x contains at least one of such neighborhoods.

J.4.1.2 Compact-Open Topology

The *compact-open topology* is an important topological construction for function spaces. It is defined on any space $X@_{\mathbf{Top}}Y$ of continuous functions $f : X \to Y$ between two topologic spaces X, Y. It is the topology generated by the sets $\langle K; U \rangle := \{f \in X@_{\mathbf{Top}}Y | f(K) \subset U\}$, where $K \subset X$ is quasi-compact and $U \subset Y$ is open. Denote by E the evaluation map $X@Y \times X \to Y : (f, x) \mapsto f(x)$.

A *locally compact* topological space X is a topological space that has a base of quasi-compact neighborhoods for every point $x \in X$. For example, metric spaces are locally compact.

Theorem 92 (Exponential Correspondence) *If X is a locally compact Hausdorff space, and Y and Z are topological spaces, then a map $g : Z \to X@Y$ is continuous (for the compact-open topology on $X@Y$) iff $E \circ (g \times Id_X) : Z \times X \to Y$ is continuous.*

Theorem 93 (Exponential Law) *If X is locally compact Hausdorff, Z is Hausdorff, and Y is a topological space, then the function $\epsilon : Z@(X@Y) \to (Z \times X)@Y$ defined by $\epsilon(g) = E \circ (g \times Id_X)$ is a homeomorphism.*

Remark 35 In particular, this theorem proves that the functor $Z \mapsto Z \times X$ is left adjoint to the functor $Y \mapsto X@Y$ (see Appendix Section G.2.1).

Theorem 94 *If X is a compact Hausdorff space and Y is metrized by a metric d, then $X@Y$ is metrized by the metric d^* defined by*

$$d^*(f, g) = \sup\{d(f(x), g(x)) | x \in X\}$$

J.4.2 Topological Categories

Intuitively, a topological[3] category is a category that is also a topological space. This is a concept which appears in the theory of internal categories, i.e., categories that are "internal" to a given category. Our case is a category that is internal to the category **Top** of topological spaces. Internal categories are simple constructions as follows:

Definition 204 *Given a category $\mathcal{E}$ with fiber products (also called* pullbacks*), an* internal category in $\mathcal{E}$ *is a pair $(\mathbf{C_0}, \mathbf{C_1})$ of objects (the first being called "object of objects", the second "object of morphisms", together with four arrows in $\mathcal{E}$:*

1. *two morphisms for category domain (d) an codomain (c)*

$$d : \mathbf{C_1} \to \mathbf{C_0}, \mathbf{c} : \mathbf{C_1} \to \mathbf{C_0}$$

2. *one identities morphism*

$$e : \mathbf{C_0} \to \mathbf{C_1}$$

simulating the identification of objects and identity morphisms,

3. *and an arrow*

$$m : \mathbf{C_1} \times_{\mathbf{C_0}} \mathbf{C_1} \to \mathbf{C_1}$$

for composition of "composable" morphisms (having connecting domains and codomains). This fiber product relates to the domain and codomain morphisms d, c.

[3] There is also a second understanding of this concept which we shall not discuss here. It has nothing to do with the one we deal with.

These morphisms are subject to a number of rules that are the diagrammatic restatement of the definition of a category:

1. Associativity, the following diagram commutes:

$$\begin{array}{ccc} \mathbf{C_1} \times_{\mathbf{C_0}} \mathbf{C_1} \times_{\mathbf{C_0}} \mathbf{C_1} & \xrightarrow{Id \times m} & \mathbf{C_1} \times_{\mathbf{C_0}} \mathbf{C_1} \\ \downarrow{\scriptstyle m \times Id} & & \downarrow{\scriptstyle m} \\ \mathbf{C_1} \times_{\mathbf{C_0}} \mathbf{C_1} & \xrightarrow{m} & \mathbf{C_1} \end{array}$$

2. Identity domains/codomains:
$$d \circ e = c \circ e = Id_{\mathbf{C_0}}$$

3. Domains and codomains of compositions:
$$d \circ m = d \circ pr_2, c \circ m = c \circ pr_1$$

4. Identity neutrality
$$m \circ (Id_{\mathbf{C_1}} \times e) = pr_1, m \circ (e \times Id_{\mathbf{C_1}}) = pr_2.$$

If two internal categories $(\mathbf{C_0}, \mathbf{C_1})$, $(\mathbf{D_0}, \mathbf{D_1})$ are given, an *internal functor* $f : (\mathbf{C_0}, \mathbf{C_1}) \to (\mathbf{D_0}, \mathbf{D_1})$ is a pair $f = (f_0 : \mathbf{C_0} \to \mathbf{D_0}, \mathbf{f_1} : \mathbf{C_1} \to \mathbf{D_1})$ of morphisms such that they define functors of the internal categories, i.e., commute with the defining morphisms e, d, c, m. This defines the *category* $Int(\mathcal{E})$ *of categories internal to* $\mathcal{E}$.

To apply such a situation to the category of topological spaces, i.e., $\mathcal{E} = \mathbf{Top}$, we need to know that **Top** has fiber products, but such objects exist, they are the set-theoretical fiber products together with the product topologies and their evident restrictions.

J.5 Complex Analysis

For an reference to complex analysis we refer to Lars V. Alfohrs' excellent book [22]. This theory is the complex number analogue to calculus on the field of real numbers. The core construction is the complex derivative of a function $f : U \to \mathbb{C}$ defined on an open set $U \subset \mathbb{C}$. If $z \in U$, the derivative of f in z is defined by

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

if this limit exist for complex h. This in particular implies that the limit is independent of the way h approaches 0 in $\mathbb{C}$, and this implies that if $f'(z)$ exists, then we have the famous *Cauchy-Riemann equation* with the real and imaginary arguments x, y in $z = x + iy$:

$$\partial_x f = -i\partial_y f.$$

And we then have $f'(z) = \partial_x f$, in which case we say that f is *analytic* or *holomorphic* in z. This differential equation is also sufficient for the existence of the complex derivative of f in z. The function f is analytic (holomorphic) if it is so on all of U.

Being analytical is a very strong property of f, as is shown by the following theorem:

Theorem 95 *Let f be analytic in an open set $U \subset \mathbb{C}$, let $z \in U$, and let $C \subset U$ be a circle curve centered in z, the disk it defines is entirely in U, then for any natural number n we have*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}$$

where $f^{(n)}$ is the nth derivative of f, especially $f^{(0)} = f$. In particular, f has all higher derivatives.

It is important to know that analytical functions are *conformal*, i.e., they locally conserve angles and therefore forms. Here is the precise fact. Suppose $\gamma :]a, b[\to U \subset \mathbb{C}$ is a differentiable curve, defined on an open neighborhood $]a, b[\subset \mathbb{R}$ containing 0, with $\gamma(0) = z$ and $T\gamma(0) = t$, and let $f : U \to \mathbb{C}$ be analytic with $f'(z) \neq 0$, then the tangent w of $f \circ \gamma$ at 0 is $w = f'(z)t$ (product of complex numbers), i.e., the argument of w is increased by the argument of $f'(z)$, independently of the t's direction. In particular, the f-image of any orthogonal basis of the tangent space Tz is again an orthogonal basis for $f'(z) \neq 0$.

Although con formality is a seemingly strong property of a function, the Riemann mapping theorem allows us to map any two simply connected open sets $U, V \subsetneq \mathbb{C}$ bijectively onto each other, more precisely:

Theorem 96 (Riemann Mapping Theorem) *Let $U \subsetneq \mathbb{C}$ be a proper simply connected open set and $z \in U$, then there exists a unique analytic function f on U that maps U bijectively onto the unit disk around 0, and such that $f(z) = 0$ and $f'(z) \in \mathbb{R}_+$ are fixed.*

Observe that the size of $f'(z)$ is not specified. For our applications to world-sheets of complex time (see Section 78.2.1), we shall impose non-automatic conditions on the derivatives of associated analytic functions. The Riemann Mapping Theorem is very strong for it connects any two simply connected proper regions in $\mathbb{C}$ by conformal mappings, but the size of the derivatives is not further controllable and is of course a function of the sizes and shapes of involved regions.

J.6 Differentiable Manifolds

Differentiable Manifolds are differentiable space structures that locally look like an open set of $\mathbb{R}^n$. Here is the definition.

Definition 205 *Let X be a topological space and n, r natural numbers. A n-dimensional $\mathcal{C}^r$-atlas for X is given by*

1. *A family A of $(f_\kappa : U_\kappa \to X)_{\kappa \in K}$ of homeomorphisms on open sets $U_\kappa \subset \mathbb{R}^n$ onto their images $Im(f_\kappa) =: X_\kappa \subset X$ that are open in X (called* charts *of the atlas) such that*
2. *$\bigcup_\kappa X_\kappa = X$ (i.e., the family $(X_\kappa)_K$ is a covering of X), and*
3. *for any pair $\kappa, \lambda \in K$ with the (automatically) open $X_\kappa \cap X_\lambda \neq \emptyset$, the bijection*

$$X_{\kappa,\lambda} \to X_{\lambda,\kappa}$$

induced by the charts on the inverse images $X_{\kappa,\lambda} = f_\kappa^{-1}(X_\kappa \cap X_\lambda), X_{\lambda,\kappa} = f_\lambda^{-1}(X_\kappa \cap X_\lambda)$ of the intersection $X_\kappa \cap X_\lambda$ is a diffeomorphism in $\mathcal{C}^r$.

Definition 206 *Two atlases A, B for X are called* equivalent *is their direct sum $A \sqcup B$ is also an atlas.*

Equivalence of atlases for X is evidently an equivalence relation. Therefore we may define

Definition 207 *Given a topological space X, an* n-dimensional $\mathcal{C}^r$-differentiable manifold *structure on X is an equivalence class of n-dimensional $\mathcal{C}^r$-atlases for X.*

In short, a differentiable manifold is a topological space X together with a representative of an atlas class. Observe that the characteristic difference between differentiable manifolds and global compositions (see [714, 13]) is that for global compositions the nerve of its covering by an atlas is fixed, it cannot be refined ad libitum like for differentiable manifolds, but see our corresponding remarks in Sections 13.1 & 19.1. $\mathcal{C}^r$-differentiable manifolds define a category Man^r whose morphisms are defined as follows[4].

[4] The category Man^r can be generalized to Banach spaces instead of $\mathbb{R}^n$, i.e., complete topological vector spaces that are defined by a norm. But we don't need this generality except for some calculus of variations.

Definition 208 *If X and Y are $\mathcal{C}^r$-differentiable manifolds, a morphism $f : X \to Y$ is a set map $f : X \to Y$ such that for every $x \in X$, and for every chart $g_\mu : V_\mu \to Y$ of Y with $f(x) \in Y_\mu$, there is a chart $f_\kappa : U_\kappa \to X$ with $f(X_\kappa) \subset Y_\mu$ and such that the corresponding map $U_\kappa \to V_\mu$ (between open sets in real spaces) is $\mathcal{C}^r$-differentiable.*

J.6.1 Manifolds with Boundary

For the Stokes Theorem we need also define compact manifolds that have a "boundary". Here is the definition:

Definition 209 *A* compact manifold with boundary *is a triple (X, X_0, BdX_0) where*

1. *X is a n-dimensional manifold,*
2. *X_0 is an open submanifold of X,*
3. *the closure of X_0 is compact,*
4. *BdX_0 is the topological boundary of X_0 (closure of X_0 minus (interior of) X_0) and a $n-1$-dimensional submanifold of X.*

The well-known Möbius strip, including its boundary line, is a famous two-dimensional manifold with boundary. The n-dimensional standard simplex Δ_n or the cube I^n are other examples, the faces being their boundary.

J.6.2 The Tangent Manifold

Locally, an n-dimensional manifold X looks like an open set in $\mathbb{R}^n$. In particular, if $m \in X$ is a point and $f_\kappa : U_\kappa \to X$ a chart, the corresponding point $x = f_\kappa^{-1}(m) \in \mathbb{R}^n$ allows to consider the "tangent"vectors at x, which we denote as pairs $t_x = (x, t), t \in \mathbb{R}^n$. This set ${}^fT_x$ is a function of the selected chart. But for manifolds in Man^r, $r \geqslant 1$, we can create a chart-independent tangent concept as follows: Take the disjoint union $\coprod_f {}^fT_x$ and consider the relation $(x,t) \sim (y,s)$ for charts f, g around m iff the Jacobi matrix at x of the isomorphism $g^{-1} \circ f$ that is defined on the intersection of the two charts maps t to s. This is an equivalence relation and the *tangent space of X at m* is defined to be the set of equivalence classes $T_mX = \coprod_f {}^fT_x/\sim$; it is obviously also a real vector space of dimension n. The tangent manifold TX is the disjoint union $TX = \coprod_{m\in X} T_mX$ with the $2n$-dimensional manifold structure that is given by the evident charts $Tf : U \times \mathbb{R}^n \to TX$, the transition morphisms being the original ones in the first component, while the second component is transformed by the Jacobian. We have the base point projection morphism $p_X : TX \to X$ sending tangent (m, t) to m.

The tangent construction is functorial, i.e., if $f : X \to Y$ is a morphism of manifolds, then we have a corresponding tangent morphism $Tf : TX \to TY$ which commutes with the projections, i.e., the diagram

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

is commutative. On the tangent spaces it is defined by the Jacobian of f on its charts. The functor $T : X \mapsto TX$ is called the *tangent functor*, and the projection $p_X : TX \to X$ defines a natural transformation $p : T \to Id_{Man^r}$. Tangent morphisms are vector bundle morphisms: they are linear on every tangent space T_mX.

Sorite 17 *All manifolds in this sorite are in Man^r, $r \geqslant 1$.*

1. *If X, Y are two manifolds, then $T(X \times Y) \tilde{\to} TX \times TY$.*
2. *In particular, for any point $(m, n) \in X \times Y$, we have $T_{(m,n)}X \times Y \tilde{\to} T_mX \oplus T_nY$.*

J.7 Tensor Fields

On n-dimensional vector space T and for every pair r, s of natural numbers, we consider the vector space of multilinear forms $T^r_s(T)$, this by definition the vector space of multilinear forms $t : T^* \times T^* \times \dots T^* \times T \times T \times \dots T \to \mathbb{R}$, the first r factors being the linear dual space T^* and the last s factors being T. By the universal property of tensor products, this is also the linear dual of $(T^*)^{r\otimes} \otimes T^{s\otimes}$. Denote this n^{r+s}-dimensional vector space by T^r_s, and therefore call its elements tensors, *r times contravriant and s times covariant.*

J.7.1 Alternating Tensors

Within T^0_s, we have the subspace $\Lambda^s T$ of alternating tensors, i.e., by definition those tensors $t : T^{s\otimes} \to \mathbb{R}$ such that for any permutation $\pi \in \mathfrak{S}_s$, we have $t(x_{\pi(1)}, x_{\pi(2)}, \dots x_{\pi(s)}) = sgn(\pi)t(x_1, x_2, \dots x_s)$. The tensor product of two alternating tensors is not alternating in general, but there is corresponding construction, namely the wedge product. To this end, one needs an auxiliary construction that is defined for any covariant tensor $t \in T^0_s$:

$$Alt(s)(x_1, x_2, \dots x_s) = \frac{1}{s!} \sum_{\pi \in \mathfrak{S}_s} sgn(\pi) s(x_{\pi(1)}, x_{\pi(2)}, \dots x_{\pi(s)}).$$

$Alt(s)$ is an alternating tensor. The *wedge product* of two alternating tensors v, w or degree r, s, respectively, is defined by

$$v \wedge w = \frac{(r+s)!}{r!s!} Alt(v \otimes w).$$

This new alternating tensor construction is associative, distributive, bilinear, and $w \wedge v = (-1)^{rs} v \wedge w$. This construction entails the calculation of the dimension of $\Lambda^s T$:

Lemma 114 *If $v_1, v_2, \dots v_k$ is a basis of T, and if $v_1^*, v_2^*, \dots v_k^*$ is the dual basis, then a basis of $\Lambda^s T$ is given by the sequence of all wedge products*

$$v^*_{i_1} \wedge v^*_{i_1} \wedge \dots v^*_{i_s}$$

with increasing indices $i_1 < i_2 < \dots i_s$. And this implies $dim(\Lambda^s T) = \binom{k}{s}$, in particular $dim(\Lambda^k T) = 1$.

Proposition 136 *If $dim(T) = k$, then for a basis $v_1^*, v_2^*, \dots v_k^*$ the base-change on $\Lambda^k T$ is covered by the following formula. If $w_i = \sum_i a_{ij} v_j, i = 1, 2 \dots k$, are any k vectors in T, then*

$$w_1 \wedge w_2 \wedge \dots w_k = det(a_{ij}) v_1 \wedge v_2 \wedge \dots v_k.$$

This result splits the bases of $\Lambda^k T$ into two classes: those which have a positive transition determinant, and those having a negative determinant. The choice of a determinant class is called *orientation* on T. On $\mathbb{R}^k$, the *usual orientation* is defined by the canonical basis $e_1, e_2, \dots e_k$.

J.7.2 Tangent Tensors

For the tangent space $T = T_m X$ of a n-dimensional manifold X, the disjoint union of the tensors $T^r_{s\,m}X := (T_m X)^r_s$ of the tangent spaces, $T^r_s X = \coprod_{m \in X} T^r_{s\,m} X$, can be given a vector bundle structure whose charts are the evident injections $T^r_s f : U \times (\mathbb{R}^n)^r_s \to T^r_s X$. The gluing isomorphisms on these charts are the canonical multilinear extensions of the isomorphisms (!) on the tangent charts.

Definition 210 *Given a differentiable manifold $X \in Man^t$, a* tensor field of type $\binom{r}{s}$ *is a $\mathcal{C}^t$-section $t : X \rightarrowtail T^r_s X$. A* differential s-form *is a tensor field of type $\binom{0}{s}$. A* vector field *is a tensor field of type $\binom{1}{0}$, where one uses the canonical isomorphism $T \xrightarrow{\sim} T^{**}$ for finite dimensional vector spaces to identify a vector field with a section $X \rightarrowtail TX$. Denote by $\mathcal{T}^r_s(X)$ the set of tensor fields of type $\binom{r}{s}$. The set of vector fields over X is denoted by $\mathcal{V}(X)$.*

Tensor fields of a given type can be added pointwise, and also be multiplied pointwise by scalar functions, i.e., manifold morphisms $f : X \to \mathbb{R}$. Denote the ring of these functions by $\mathcal{F}X$ (and also denote it by $\mathcal{T}^0_0(X)$). If $f \in \mathcal{T}^r_s(X), g \in \mathcal{T}^{r'}_{s'}(X)$, then their pointwise product is in $\mathcal{T}^{r+r'}_{s+s'}(X)$, we denote it by $f \otimes g$ in view of the tensor product formalism we recalled above. With these operations, the direct sum $\mathcal{T}X = \bigoplus_{r,s\geqslant 0} \mathcal{T}^r_s(X)$ is a bigraded algebra over the ring $\mathcal{F}X$ of functions on X, the *tensor algebra of* X.

If $f \in \mathcal{F}X$, then we have $Tf : TX \to T\mathbb{R} = \mathbb{R}^2$. If we look at the second component of Tf, we get a function $df : TX \to T\mathbb{R} = \mathbb{R}$, i.e., a 1-form, the *differential of* f. Given a vector field V on X, we may then define a new function $L_V f(m) = df_m(V(m))$ in $\mathcal{F}X$. This function is called the *Lie derivative of* f *with respect to* V. Here are the characteristic properties of this assignment:

Proposition 137 *The map $L_V : \mathcal{F}X \to \mathcal{F}X$ is a derivation on the function ring, i.e., it is $\mathbb{R}$-linear and $L_V(fg) = (L_V f)g + f(L_V g)$. Moreover, $L_V c = 0$ for a constant function c.*

Denote by $Der(X)$ the real vector space of derivations on X. This result defines a function $L : \mathcal{V}(X) \to Der(X) : V \mapsto L_V$.

Theorem 97 *The Lie derivative function*

$$L : \mathcal{V}(X) \to Der(X)$$

is a linear isomorphism of real vector spaces.

This means that derivations and vector fields are essentially the same thing. This fact is very important in musical performance theory where for important performance operators, the Lie derivative is taken as a function of a performance field that acts upon performance weights, generating a new performance field, but see Section 39.7.

On a n-dimensional differentiable manifold X, an orientation is defined by a section $D : X \rightarrowtail \Lambda^n X$ that has non-vanishing values D_m in all the one-dimensional spaces $\Lambda^n T_m X$. A manifold need not have any orientation, it can be *non-orientable*; an example is the Möbius strip (without its boundary line).

J.8 Stokes' Theorem

This theorem connects integrals over differential forms on a compact manifold with boundary with integrals of the derivative of such forms on the manifold's boundary.

The first concept is the integral of a differential form. We cannot recapitulate the extensive setup here and refer to [617] for a precise reference. The essential steps are these: First consider the integral $\int_{I^n} f$ of a sufficiently tame function f on a cube I^n, then take the integral $\int_{I^n} w$ on a differential form $w = f.d$ on the manifold I^n with respect to a basis $d \in \Lambda^n I^n$, which is by definition $\int_{I^n} f$ for that function f of points in I^n. Next, one takes a singular cube[5], i.e., a morphism $c : I^n \to X$ into an oriented n-dimensional manifold X with orientation form d. One integrates a form $w = f.d$ by taking the inverse image c^*w of w, and then integrates this one, i.e., $\int_c w = \int_{I^n} c^*w$. Next, one takes a general oriented manifold and calculates integrals using partitions of unity, which enable reduction to the singular cube situation.

We next need to make precise the manifolds with boundary and orientation needed for Stokes.

Definition 211 *An* oriented compact manifold with boundary *is a compact manifold with boundary (see Section J.6.1) (X, X_0, BdX_0) in which X is oriented, and for each $b \in BdX_0$, there is a positively oriented chart $f : U \to X$ with the following properties (see Figure J.2):*

1. *There is a submanifold chart $g : U' \to BdX_0$ with $U \subset U' \times I \subset \mathbb{R}^{n-1} \times \mathbb{R}$, b being represented by* 0,
2. $f^{-1}(f(U) \cap BdX_0) = U' \times \{0\}$,
3. $f^{-1}(f(U) \cap X_0) \subset U' \times]0, 1[$.

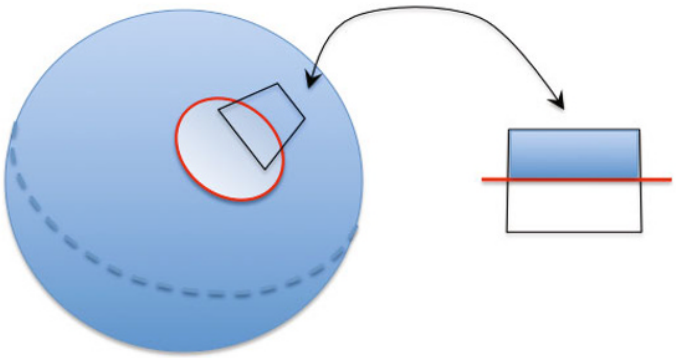


Fig. J.2. The opened spherical surface manifold with a boundary chart construction.

The last ingredient is about the differential of a differential form: We already know the differential df of a function, which is a 1-form. The general definition is s follows: The idea is to define the differential $dw \in \Lambda^{k+1}X$ of a k-form $w \in \Lambda^k X$, $dim(X) = n$, on charts, to show that the definition is independent of the chart, and then to glue the local definitions to a global one. If we have local coordinates $x_1, x_2, \ldots x_n$ on a chart, then we have

$$w = \sum_{i_1<i_2<\ldots i_k} w_{i_1,i_2,\ldots i_k} dx_{i_1} \wedge dx_{i_2} \wedge \ldots dx_{i_k},$$

and we define the $k+1$ form, the *differential of* w by

$$dw = \sum_{i_1,i_2,\ldots i_k} dw_{i_1,i_2,\ldots i_k} dx_{i_1} \wedge dx_{i_2} \wedge \ldots dx_{i_k} + \sum_{i_1,i_2,\ldots i_k} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} w_{i_1<i_2<\ldots i_k} dx_j \wedge dx_{i_1} \wedge dx_{i_2} \wedge \ldots dx_{i_k}.$$

And here is the basic theorem about the differential operator:

Theorem 98 *The differential operator $d : \Lambda^k X \to \Lambda^{k+1} X$ is a $\mathbb{R}$-linear map with*

1. $d^2 = 0$,
2. *for a k-form w and a l-form y we have*

$$d(w \wedge y) = dw \wedge y + (-1)^{kl} w \wedge dy,$$

if $f : X \to Y$ is a morphism of manifolds and $w \in \Lambda^k Y$, then

$$f^* dw = df^* w.$$

The first statement implies that we have a cochain complex (see Example 139 in Section J.13) of differential forms which defines the associated cohomology modules (in fact real vector spaces).

Theorem 99 (Stokes' Theorem) *If $w \in \Lambda^{k-1}X$ and if (X, X_0, BdX_0) is an oriented compact k-dimensional manifold with boundary, and denote by $i : BdX_0 \to X$ the inclusion, then we have*

$$\int_{BdX_0} i^* w = \int_{\overline{X_0}} dw$$

where $\overline{X_0}$ is the topological closure of X_0.

[5] See this Appendix, Section J.12.1 for this concept.

J.9 Calculus of Variations

The calculus of variations deals with the calculus of local extrema of functionals, which are defined as being numerical functions J (usually real-valued) on spaces of continuous functions $f : [a, b] \to \mathbb{R}^n$ on a closed interval $[a, b] \subset \mathbb{R}$. The space $\mathcal{C}^0_n[a, b]$ of $\mathbb{R}^n$-valued continuous functions is a Banach space with norm $\| \ \|$ (see Appendix Section I.1.2 for norms and Banach spaces), defined by

$$_0\|f\| = max\{\|f(x)\| \mid x \in [a, b]\}.$$

and if we restrict to the subspace $\mathcal{C}^1_n[a, b] \subset \mathcal{C}^0_n[a, b]$ of $\mathcal{C}^1$-functions, we have the norm[6]

$$_1\|f\| = max(_0\|f\|, \ _0\|f'\|).$$

With these notations, we define extremal values:

Definition 212 *A function $f \in \mathcal{C}^0_n[a, b]$ is a* strong maximum(minimum) of a functional J *on $\mathcal{C}^0_n[a, b]$ if the values $J(g) - J(f) \leqslant 0 (J(g) - J(f) \geqslant 0)$ for all $g \in U(f)$ for some neighborhood $U(f) \subset \mathcal{C}^0_n[a, b]$ of f. A function $f \in \mathcal{C}^1_n[a, b]$ is a* weak maximum(minimum) of a functional J *on $\mathcal{C}^1[a, b]$ if the values $J(g) - J(f) \leqslant 0 (J(g) - J(f) \geqslant 0)$ for all $g \in U(f)$ for some neighborhood $U(f) \subset \mathcal{C}^1_n[a, b]$ of f.*

The following *Fundamental Lemma of Calculus of Variations* is useful for finding extremal functions for functionals that involve integrals.

Lemma 115 *Let $f \in \mathcal{C}^k[a, b]$, and suppose that we have*

$$\int_a^b f(x)h(x) = 0$$

for every $h \in \mathcal{C}^k[a, b]$ with $h(a) = h(b) = 0$. Then $f = 0$.

Its proof is an easy exercise (choose a special function $h(x) = f(x)(x - a)(b - x)$). The lemma is useful for the calculation of Lagrangians in theoretical physics (see Section K.1.1).

J.10 Partial Differential Equations

Quasi-linear first order partial differential equations have been discussed in Appendix Section I.6. Here, we need to look at a special type of second order partial differential equations: $\Delta w + U = 0$, where w and U are functions of two real variables x, y. This one is known as Poisson's equation with the Laplace operator $\Delta = \partial^2_x + \partial^2_y$. This equation appears in the performance world-sheet discussion in Section 78.

We shall discuss here the solution of such an equation for $0 \leqslant x, y \leqslant 1$, the more general frame $0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b$ being a straightforward generalization. The solution is given in several steps of special cases. First, we solve the equation for zero potential (a Laplace equation). Then, we solve the equation with $w|\partial I^2 = 0$, and then we add these two special solutions and get the general one.

To begin with, in [768, §7.1], an explicit solution of the Poisson equation $\Delta w + U = 0$ in a special case is given[7]. The potential is supposed to be zero. The function w is supposed to vanish on the boundary ∂I^2 except for the upper line $y = 1$ where the function $w_1(x)$ is given. The solution is

$$w(x, y) = \int_{\xi=0}^{1} w_1(\xi) G^{y=1}(x, y|\xi), \tag{J.2}$$

[6] Attention: the usual index notation conflicts with the absolute or 1-norm described in Section I.1.2, we therefore write the index to the left of the norm symbol.

[7] We are grateful to theoretical physicist Emilio Fiordilino for his help to understand the Green function in our situation.

where ([768, formula 7.1.8]) the Green function is defined by

$$G^{y=1}(x,y|\xi) = 2\sum_{n=0}^{\infty} \frac{\sinh(\pi n y)}{\sinh(\pi n)} \sin(\pi n x)\sin(\pi n \xi).$$

The fact that $w(x,1) = w_1(x)$ relies on the standard integral expression of Fourier series coefficients and on the classical orthogonality relations.

Next, we find a solution of $\Delta w + U = 0$ with $w|\partial I^2 = 0$. In [768, §7.1, equation (7.1.12)], we are given this solution:

$$w(x,y) = \int_{I^2} G(x,y|\xi,\eta)U(\xi,\eta)$$

with the Green function

$$G(x,y|\xi,\eta) = 8\sum_{n=0}^{\infty} \frac{\sin(\pi n x)\sin(\pi n \xi)}{n\sinh(\pi n)} H_n(y,\eta)$$

where

$$H_n(y,\eta) = \begin{cases} \sinh(p_n\eta)\sinh(p_n(1-y)) & \text{for } 1 \geqslant y > \eta \geqslant 0, \\ \sinh(p_n y)\sinh(p_n(1-\eta)) & \text{for } 1 \geqslant \eta \geqslant y \geqslant 0. \end{cases}$$

The general case can be deduced in the following way. To begin with, the given solution can be found also for w that vanishes on all but one side of I^2, simply switch x and y or exchange 1 and 0. Then, if the four solutions $w^i, i = 1,2,3,4$, are found for the four situations $y = 1, y = 0, x = 1, x = 0$ and the potential $U/4$, their sum $w = w^1 + w^2 + w^3 + w^4$ solves the original equation $\Delta w + U = 0$. But this w is still special in that its values $w(x,y)$ vanish for all four corner points $(0,0),(0,1),(1,0),(1,1)$.

To solve the general case, we need a function $c(x,y)$ that solves the Laplace equation $\Delta c = 0$ and has arbitrary values on the four corner points. Then the general w can be corrected to $w - c$ and we may apply the given solution for this function that vanishes on the four corner points.

To begin with, we find an affine function $a(x,y) = rx + sy + t$ such that $a(0,0) = w(0,0), a(1,0) = w(1,0), a(0,1) = w(0,1)$. With the subtraction $w - a$, and observing that $\Delta a = 0$, we reduce w to a function that has all corner values equal to zero except possibly $w(1,1)$. Take the function $p(x,y) = w(1,1)xy$. Evidently $\Delta p = 0$, and p vanishes for the three corners of I^2 except for corner $(1,1)$ where its value is $w(1,1)$. Therefore $w - p$ has vanishing values on all four corners.

J.10.1 Explicit Calculation

We calculate explicitly the solution with the previous method. Recapitulate the idea of solving the Poisson equation $\Delta w + U = 0$:

1. Build $w_1 = w + l$, $l(x,y) = ax + by + c$ such that $w_1(0,0) = w_1(1,0) = w_1(0,1) = 0$ and $\Delta l = 0$,
2. Build $w_2 = w_1 - q, q(x,y) = dxy$ such that w_2 vanishes on all four angles of I^2, and $\Delta q = 0$.
3. Then solve two partial problems: (1) $w_{2,1} + U = 0$ with $w_{2,1}|\partial I^2 = 0$, and (2) $\Delta w^i_{2,2} = 0$ with $w^i_{2,2}|\partial I^2_i = w_2|\partial I^2_i$ and $w^i_{2,2}|\partial I^2_j = 0$ for $j \neq i$.
4. Then $w_2 = w_{2,1} + \sum_{i=1}^4 w^i_{2,2}$ solves the equation $\Delta w_2 + U = 0$, and the original function is $w = w_2 - Q, Q = l - q$.

The first two steps yield this representation:

$$\begin{aligned} a &= w(0,0) - w(1,0) \\ b &= w(0,0) - w(0,1) \\ c &= -w(0,0) \\ d &= w(1,1) + w(0,0) - w(1,0) - w(0,1) \\ w_2(x,y) &= w(x,y) + Q(x,y) \\ &= w(x,y) + ax + by + c - dxy \end{aligned}$$

Next, we have to define the Green functions for the four sides of the second part of the third step. For side $y = 1$, the Green function has been defined above by the expression $G^{y=1}(x, y|\xi)$. For $x = 1$ we have to exchange the roles of x and y, whence $G^{x=1}(x, y|\eta) = G^{y=1}(y, x|\eta)$. For $y = 0$ we switch from y to $1 - y$: $G^{y=0}(x, y|\xi) = G^{y=1}(x, 1 - y|\xi)$, and for $x = 0$ we get $G^{x=0}(x, y|\eta) = G^{x=1}(1 - x, y|\eta)$.

And here are the functions on the four sides of ∂I^2:

$$\begin{aligned} w_2|_{y=0} &= w|_{y=0} + ax + c \\ w_2|_{x=0} &= w|_{x=0} + by + c \\ w_2|_{y=1} &= w|_{y=1} + ex - w(0,1) \\ w_2|_{x=1} &= w|_{x=1} + fy - w(1,0) \end{aligned}$$

with $e = w(0,1) - w(1,1), f = w(1,0) - w(1,1)$.

With this data, we can now calculate the solutions in the second part of step three:

$$\begin{aligned} w_{2,1}(x,y) &= \int_{I^2} G(x,y|\xi,\eta)U(\xi,\eta) \\ w_{2,2}^{y=1}(x,y) &= \int_{\xi=0}^{1} w_2|_{y=1} G^{y=1}(x,y|\xi) \\ &= \int_{\xi} w|_{y=1} G^{y=1}(x,y|\xi) + e\int_{\xi} \xi G^{y=1}(x,y|\xi) - w(0,1)\int_{\xi} G^{y=1}(x,y|\xi) \\ w_{2,2}^{y=0}(x,y) &= \int_{\xi=0}^{1} w_2|_{y=0} G^{y=0}(x,y|\xi) \\ &= \int_{\xi} w|_{y=0} G^{y=0}(x,y|\xi) + a\int_{\xi} \xi G^{y=0}(x,y|\xi) + c\int_{\xi} G^{y=0}(x,y|\xi) \\ w_{2,2}^{x=1}(x,y) &= \int_{\eta=0}^{1} w_2|_{x=1} G^{x=1}(x,y|\eta) \\ &= \int_{\eta} w|_{x=1} G^{x=1}(x,y|\eta) + f\int_{\eta} \eta G^{x=1}(x,y|\eta) - w(1,0)\int_{\eta} G^{x=1}(x,y|\eta) \\ w_{2,2}^{x=0}(x,y) &= \int_{\eta=0}^{1} w_2|_{x=0} G^{x=0}(x,y|\eta) \\ &= \int_{\eta} w|_{x=0} G^{x=0}(x,y|\eta) + b\int_{\eta} \eta G^{x=0}(x,y|\eta) + c\int_{\eta} G^{x=0}(x,y|\eta) \end{aligned}$$

This means that $w_{2,1}$ only depends linearly on the potential, $w_{2,1} = P(U)$, while the four functions $w_{2,2}^{y=1}, w_{2,2}^{y=0}, w_{2,2}^{x=1}, w_{2,2}^{x=0}$ depend in a linear way on the restriction of w to the four sides $w|y = 1, w|y = 0, w|x = 1, w|x = 0$, respectively, i.e., to the components of the boundary function ∂w. The values of the constants a, b, c, d are only a linear function of the values of w on the four angles of I^2, i.e., we have

$$w = P(U) + S(\partial w)$$

for two linear functions P, S.

J.11 Algebraic Topology

Refer also to Appendix Section H.2.

J.11.1 Homotopy Theory

Homotopy theory is the basic conceptual setup of algebraic topology. It deals with continuous curves $I \to X$, defined on the unit interval $I = [0,1] \subset \mathbb{R}$ in topological spaces X and so doing it is hoped to understand X in the best way possible. This is a special case of the Yoneda Lemma (Appendix Section G.2), where the sheaf $@X$ is used to classify X. The continuous curves $I \to X$ are not sufficient to completely understand X, but they provide us with a lot of information, and one could say that Algebraic Topology is the effort to approximate Yoneda's Lemma by a special type of morphisms (their domains being special objects that one understands well).

Definition 213 *A* topological pair *is a pair (X, A) of a topological space X, together with a subspace $A \subset X$. If $card(A) = 1$, we call the pair* a pointed topological space *(X), and the point in question is the singleton $a \in A$. If (Y, B) is a second topological pair, a morphism $f : (X, A) \to (Y, B)$ is a continuous function $f : X \to Y$ such that $f(A) \subset B$. The* category of topological pairs *is denoted by* **Top**2*.*

Homotopy is about comparing different morphisms in **Top**2:

Definition 214 *If $(X, A), (Y, B)$ are two topological pairs and if $X' \subset X$ is a subspace of X, a homotopy from a morphism $f_0 : (X, A) \to (Y, B)$ to a morphism $f_1 : (X, A) \to (Y, B)$ relative to X' is a continuous map $F : X \times I \to Y$ such that $F(?, t) : (X, A) \to (Y, B)$ is a morphism for all $t \in I$, with $F(?, t)|_{X'} = f_0|_{X'}$, all $t \in I$, $f_0 = F(?, 0)$ and $f_1 = F(?, 1)$. In symbols: $F : f_0 \simeq_{X'} f_1$. In particular, f_0 is called* null homotopic *if it is homotopic to a constant map f_1. The attribute "relative to X'" is omitted for $X' = \varnothing$.*

The symbol choice $\simeq$ is justified be this easy lemma:

Lemma 116 *The relation of homotopy among the morphisms $f : (X, A) \to (Y, B)$ relative to a fixed $X' \subset X$ is an equivalence relation.*

Definition 215 *The set of homotopy equivalence classes of morphisms $f : (X, A) \to (Y, B)$ relative to a fixed $X' \subset X$ is denoted by $[X, A; Y, B]_{X'}$. We denote by $[f]_{X'}$ the class of morphism f. If $X' = \varnothing$, we omit the index X'.*

Homotopic maps can be composed:

Theorem 100 *If $f_0 \simeq_{X'} f_1$ in $[X, A; Y, B]_{X'}$ and $g_0 \simeq_{Y'} g_1$ in $[Y, B; Z, C]_{Y'}$ with $f_0(X') \subset Y'$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$ in $[X, A; Z, C]_{X'}$.*

This theorem shows that we have categories of homotopy classes of topological pair morphisms, so-called *homotopy categories*. In particular, we have the homotopy subcategory of pointed topological spaces or the homotopy category of topological spaces (pairs with empty subsets). Clearly the map $f \mapsto [f]$ is a functor from topological pairs to homotopy categories. A diagram of topological pairs that becomes commutative after passing to the homotopy classes is called *homotopy commutative*. A pair morphism that becomes an isomorphism after passage to homotopy is called a *homotopy equivalence*, and a pair morphism that becomes the inverse to another morphism after passage to homotopy is called a *homotopy inverse*. Pairs that become isomorphic after passage to homotopy are said to have the *same homotopy type*.

Here is the construction of a terminal object in the homotopy category: A topological space X is called *contractible* if its identity morphism is homotopic to a constant endomorphism of X. A homotopy from the identity Id_X to a constant map at $x_0 \in X$ is called a *contraction of X to x_0*.

Lemma 117 *Any two maps of an arbitrary topological space X to a contractible space are homotopic, i.e., there is (always) one and only one morphism in the homotopy category to a contractible space. In particular, any two endomorphisms of a contractible space are homotopic.*

This means that contractible spaces define homotopy terminal objects. And here its an easy consequence:

Theorem 101 *A space is contractible iff its homotopy type is that of a one-point space. In particular, any two contractible spaces have the same homotopy type, and any morphism between contractible spaces is a homotopy equivalence (isomorphism).*

J.11.2 The Fundamental Group(oid)

The fundamental groupoid $\mathcal{P}(X)$ of a topological space X is a groupoid, i.e., by definition, a small category whose morphisms are all isomorphisms. Here is its construction: It has as objects the elements of X, and for any two such elements x, y, a morphism $f : x \to y$ is the homotopy class of a map $f : I \to X$ with $f(0) = x, f(1) = y$, the homotopy being relative to the boundary $\partial I = \{0, 1\}$. The identity on a point x is represented by the class of the constant map $\epsilon_x : I \to X$ in x. The composition $[g] \circ [f]$ of two morphisms $[f : I \to X], [g : I \to X]$ with $f(1) = g(0)$ is $[g * f]$, where

$$g * f(t) = f(2t) \text{ for } 0 \leqslant t \leqslant 1/2, \text{ and } g * f(t) = g(2t - 1) \text{ for } 1/2 \leqslant t \leqslant 1.$$

A proof of the category-theoretical properties of this construction is found in [993, Chap. 1, Sec. 7].

This specializes to the fundamental group construction:

Definition 216 *Let (X, x_0)be a pointed topological space, then the* fundamental group of X at x_0 *is the group $\pi(X, x_0) = x_0 @_{\mathcal{P}(X)} x_0$ of automorphisms of the fundamental groupoid of X at x_0.*

This defines the *fundamental group functor* π : **Top**. $\to$ **Grp** from the category **Top**. of pointed topological spaces to the category **Grp** of groups.

J.12 Homology

Homology is a vast field that can be understood from different points of view. An evident historical one is that homology generalizes the concept of a hole in a topological space. From a more category-theoretical point of view, it is the theory which quantifies the Yoneda philosophy, namely the idea of understanding/classifying an object X (here: a topological space) by considering classes of morphisms $D \to X$ from special simple objects D, hoping to extract characteristic information about X via these special arrows. Let us recall that in set theory, classifying a set X can be done by the maps $D \to X$, where D is a singleton, since the cardinality of X is given by the cardinality of the set $D @_{\mathbf{Ens}} X$.

The formalism of homology starts from a *chain complex* over a commutative ring R. This is by definition a sequence

$$C = \ldots C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \ldots$$

of r-modules $C_q, q \in \mathbb{Z}$ and module "differential" homomorphisms ∂_q such that for all q, $\partial_q \circ \partial_{q+1} = 0$. The elements of C_q are called *q-chains*, and C is called nonnegative if $C_q = 0$ for $q < 0$. If all chain modules are free over R, C is called a *free chain complex*.

Observe that because of $\partial_q \circ \partial_{q+1} = 0$, we have the inclusion $Im(\partial_{q+1}) \subset Ker(\partial_q)$. The quotient module $H_q(C) = Ker(\partial_q)/Im(\partial_{q+1})$ is called the *qth homology module of C*. The *complex $Z(C)$ of cycles of C* is the subcomplex $Z(C) = (Ker(\partial_q))_q$ of C with the zero homomorphisms everywhere. The *complex $B(C)$ of boundaries of C* is the subcomplex $(Im(\partial_{q+1}))_q$ of C with the zero homomorphisms everywhere. We then have $B(C) \subset Z(C)$, and the quotient complex $H(C) = Z(C)/B(C)$, i.e., $H_q(C) = Z_q(C)/B_q(C)$ for all q.

This construction is functorial in the following sense. Consider the category of chain complexes $Chain_R$ over R. Its objects are the chain complexes over R while the morphisms $f : C \to C'$ are the sequences $f = (f_q)_q$ of R-module homomorphisms $f_q : C_q \to C'_q$ with commutative diagrams

$$\begin{array}{ccc} C_q & \xrightarrow{f_q} & C'_q \\ {\scriptstyle \partial_q}\downarrow & & \downarrow{\scriptstyle \partial'_q} \\ C_{q-1} & \xrightarrow{f_{q-1}} & C'_{q-1} \end{array}$$

for all q. Then we have a *homology functor* $H : Chain_R \to \mathbf{Mod}_R : C \mapsto H(C)$ to the category of R-modules.

There are several homology theories (defining chain complexes in different contexts). We shall only need a small selection here, see [635, 993] for a more complete discussion.

J.12.1 Singular Homology

This is a classical concrete theory. We denote by $|\Delta_q|$ the standard (topological) simplex space of dimension q associated with the standard simplex $\Delta_q = \{0, 1, 2, \dots q\}$, see Appendix Section H.3. For each index $i \in \Delta_q$, there is an affine embedding $e^i_q : |\Delta_{q-1}| \to |\Delta_q|$ sending the vertex e_j to e_j if $j < i$, and to e_{j+1} if $j \geqslant i$. The image of e^i_q is called the *ith face of* $|\Delta_q|$ and denoted by $|\Delta_q|^i$. Standard simplex spaces are one type of "simple" special topological spaces D mentioned above.

A second type of "simple" spaces are the q-dimensional cubes I^q, the q powers of the real unit interval I. Here we also have the face morphism: $e^{i\alpha}_q : I^{q-1} \to I^q, \alpha = +, -$, with $e^{i\alpha}_q(x_0, \dots x_{q-1})_j = x_j$ for $j < i$ and $e^{i-}_q(x_0, \dots x_{q-1})_i = 0$, $e^{i+}_q(x_0, \dots x_{q-1})_i = 1$, and $e^{i\alpha}_q(x_0, \dots x_{q-1})_j = x_{j+1}$ for $j > i$. e^{i+}_q id called the front face while e^{i-}_q id called the back face.

Given a topological space X, the q-chains for the *simplex* approach are defined starting by the basic continuous maps $c : |\Delta_q| \to X$. The simplex chain complex R-module $\Delta_R(X)_q$ for $0 \leqslant q$ is the free R-module generated by the basis of the maps c. In other words, the q-chains in $\Delta(X)_q$ are the formal linear combinations $\sum_k r_k c_k$ with $r_k \in R$ and $c_k : |\Delta_q| \to X$. The chain modules for negative q vanish by definition.

The differential homomorphism is defined as follows: Suppose that $c : |\Delta_q| \to X$ is a base element in $\Delta_R(X)_q$ for $q > 0$. Then

$$\partial_q(c) = \sum_i (-1)^i c \circ e^i_q,$$

the alternate sum of c's (restriction to its) faces. This definition is extend linearly to general q-chains.

For the cube approach, the q-chains are defined starting from the basic continuous maps $c : I^q \to X$. The cube chain complex R-module $\square_R(X)_q$ is the free R-module generated by the basis of the maps c, similarly to the above simplex approach.

The differential homomorphism is defined as follows: Suppose that $c : I^q \to X$ is a base element in $\square_R(X)_q$ for $q > 0$. Then

$$\partial_q(c) = \sum_i (-1)^i (c \circ e^{i+}_q - c \circ e^{i-}_q),$$

the alternate sum of c's (restriction to its) $+$- and $-$-faces. This definition is extend linearly to general q-chains. The important fact about the homology modules of these two approaches is this:

Theorem 102 *For a topological space X and the two singular homology approaches with simplexes and cubes we have isomorphic homology modules:*

$$H(\Delta_R(X)) \xrightarrow{\sim} H(\square_R(X)).$$

A proof can be found in [470]. While the simplex approach is standard in algebraic topology, although Jean-Pierre Serre used the cubic approach in his thesis, the cube approach is more useful for the generalized homology theory of gestures, see Chapter 63. See also [661] for the cube approach.

J.13 Cohomology

Formally speaking, cohomology is derived from a *cochain complex* over a commutative ring R. This is essentially about giving all arrows a reversed direction. But this reversal can also be induced by a chain complex, an example will be given after the definitions. A cochain complex is by definition a sequence

$$C^* = \dots C^q \xrightarrow{\partial^q} C^q \xrightarrow{\partial^{q+1}} C^{q+1} \dots$$

of r-modules $C^q, q \in \mathbb{Z}$ and module "differential" homomorphisms ∂^q such that for all q, $\partial^{q+1} \circ \partial^q = 0$. The elements of C^q are called *q-cochains*, and C^* is called nonnegative if $C^q = 0$ for $q < 0$. If all chain modules are free over R, C^* is called a *free cochain complex*.

Observe that because of $\partial^{q+1} \circ \partial^q = 0$, we have the inclusion $Im(\partial^q) \subset Ker(\partial^{q+1})$. The quotient module $H^q(C) = Ker(\partial^q)/Im(\partial^{q-1})$ is called the *qth cohomology module of C*. The *complex $Z^*(C^*)$ of* cocycles of C^* is the subcomplex $Z^*(C) = (Ker(\partial^q))_q$ of C^* with the zero homomorphisms everywhere. The *complex $B^*(C)$ of* coboundaries of C^* is the subcomplex $(Im(\partial^{q-1}))_q$ of C^* with the zero homomorphisms everywhere. We then have $B^*(C^*) \subset Z^*(C^*)$, and the quotient complex $H^*(C^*) = Z^*(C^*)/B^*(C^*)$, i.e., $H^q(C^*) = Z^q(C^*)/B^q(C^*)$ for all q.

This construction is functorial in the following sense. Consider the category of cochain complexes $Cochain_R$ over R. Its objects are the cochain complexes over R while the morphisms $f : C^* \to C'^*$ are the sequences $f = (f^q)_q$ of R-module homomorphisms $f^q : C^q \to C'^q$ with commutative diagrams

$$\begin{array}{ccc} C^q & \xrightarrow{f^q} & C'^q \\ {\scriptstyle \partial^q}\downarrow & & \downarrow{\scriptstyle \partial'^q} \\ C^{q+1} & \xrightarrow{f^{q+1}} & C'^{q+1} \end{array}$$

for all q. Then we have a *cohomology functor* $H^* : Cochain_R \to \mathbf{Mod}_R : C^* \mapsto H^*(C^*)$ to the category of R-modules.

Example 138 A generic example of a cohomology complex is to start with a chain complex $C \in Chain_R$ and to take the Hom_R functor, mapping every chain module C_q to its dual $C^q = C_q^* = Hom_R(C_q, R)$, and same for the differentials, reverting the original directions, and also keeping the vanishing of the composition of successive differential homomorphisms.

Example 139 A second example is the *de Rham cohomology* defined by the differential operator on differential forms on manifolds, as introduced in Section J.8.

There are several cohomology theories (defining chain complexes in different contexts). We shall only need a small selection here, see [635, 993] for a more complete discussion. Let us refer to the examples we have already discussed previously: Section 16.1.2, Section 19.1.1, and Appendix Section H.3.1.

Part XXII

Appendix: Complements in Physics

K

Complements on Physics

Summary. The appendix recalls some special topics in theoretical physics that are referred to in the main text. They are neither self-contained nor complete.

– Σ –

K.1 Hamilton's Variational Principle

Often physical laws are not deduced from mathematical results as theorems or corollaries, they are found without logical stringency from thought or real experiments in the framework already existing mathematical models of physical phenomena. This procedure is one of two possible approaches, as they have been described in [2, Section 19]. The second approach is more axiomatic, and it enables physicists to deduce physical laws as corollaries of metaphysical principles, i.e., of principles that are not directly present in physical reality, but which enable a deduction of physical laws. *Hamilton's variational principle* is such a metaphysical principle. It is applied in classical mechanics and in electrodynamics, but also in string theory (see Appendix K.2).

Hamilton's classical principle can be stated using the formalism of Lagrangian functions $L(t, x, \dot{x})$ of time t, space point x, and velocity $\dot{x} = dx/dt$. The typical Lagrangian for a classical point particle of mass m is $L = T - U$, where $T(\dot{x}) = \frac{m}{2}\dot{x}^2$ is the particle's kinetic energy[1] and $U(x)$ is a potential whose gradient defines forces that act upon the point particle.

The Lagrangian action is a functional of the $\mathcal{C}^1$ path function $x : t \mapsto x(t)$ of time t:

$$S(x) = \int_{t_0}^{t_1} L,$$

an integral over a time interval $[t_0, t_1]$, $t_0 < t_1$. Hamilton's principle uses the calculus of variations (see Appendix J.9) for the functional $J = S$ acting on paths $x : t \mapsto x(t)$. We consider all $\mathcal{C}^1$ paths $y : t \mapsto y(t)$ on $[t_0, t_1]$ which coincide with x on the endpoints t_0, t_1.

Principle 30 *Hamilton's action principle claims that a physical solution for such a path x must be a weak minimum of the functional S.*

In particular, the weak minimum condition can be applied to small variations $x + \varepsilon\eta$ of x defined by $\varepsilon\eta$, where η is a $\mathcal{C}^1$ path on $[t_0, t_1]$ and vanishes at the endpoints, and where ε is close to zero. The variational component $\varepsilon\eta$ is usually denoted by δx, whence the condition $S(s + \delta x) \geqslant S(x)$ in terms of variational calculus. But observe that this a local condition on x, it does not automatically imply that S is globally minimal. In particular, this means that for a fixed η, the function $\Sigma_\eta(\varepsilon) = S(x + \varepsilon\eta)$ has a local minimum

[1] The expression $\dot{x}^2$ means the scalar product $\dot{x} \cdot \dot{x}$, and we also have evident scalar products in the following products of vectorial functions.

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G. Mazzola, *The Topos of Music IV: Roots*, Computational Music Science,
https://doi.org/10.1007/978-3-319-64495-0_11

for $\varepsilon = 0$. That implies $d\Sigma_\eta/d\varepsilon = 0$ for $\varepsilon = 0$. This is the basis for the Euler-Lagrange equation for Lagrange functions.

K.1.1 Euler-Lagrange Equations for a Non-relativistic Particle

We have

$$d\Sigma_\eta/d\varepsilon\Big|_{\varepsilon=0} = \int_{t_0}^{t_1} \frac{dL}{d\varepsilon}\Big|_{\varepsilon=0} = 0$$

and, with $y = x + \varepsilon\eta$ instead of x,

$$\frac{dL}{d\varepsilon} = \frac{\partial L}{\partial y}\frac{dy}{d\varepsilon} + \frac{\partial L}{\partial \dot{y}}\frac{d\dot{y}}{d\varepsilon}$$

as t does not vary with ε. But $\frac{dy}{d\varepsilon} = \eta$ and $\frac{d\dot{y}}{d\varepsilon} = \dot{\eta}$, whence

$$\frac{dL}{d\varepsilon} = \frac{\partial L}{\partial y}\eta + \frac{\partial L}{\partial \dot{y}}\dot{\eta}$$

Using integration by parts, this implies

$$\begin{aligned} 0 = & \int_{t_0}^{t_1} \frac{dL}{d\varepsilon}\Big|_{\varepsilon=0} \\ = & \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x}\eta + \frac{\partial L}{\partial \dot{x}}\dot{\eta}\right) \\ = & \int_{t_0}^{t_1} \eta\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial \dot{x}}\eta\Big|_{t_1}^{t_0} \end{aligned}$$

But the second summand vanishes because $\eta(t_0) = \eta(t_1) = 0$. The fundamental Lemma 115 of calculus of variations (applied to every component of the vectorial function η) implies the *Euler-Lagrange equation*

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0.$$

Taking the classical Lagrange function $L(x, \dot{x}) = \frac{m}{2}\dot{x}^2 - U(x)$, we get

$$-\nabla U = m\ddot{x},$$

the classical second Newton law of mechanics, using the *nabla operator* $\nabla = (\frac{\partial}{\partial x_1}, \ldots \frac{\partial}{\partial x_n})$.

K.2 String Theory

In string theory, which is still (when this book is being written) a hypothetical theory despite its elegance and explanatory power, the elementary objects are not point particles but string particles. We have a corresponding Lagrange function and a Euler-Lagrange equation. As pointed out by Barton Zwiebach in [1159, p.74], already the non-relativistic Euler-Lagrange equation for a string shows the phenomenon of *D-branes*, i.e., the necessity of extended physical objects that are connected to a string and interact with its momentum, see also [440] for a short introduction to D-branes.

Because a string is determined by two spatial positions, horizontal x between $x = 0$ and $x = a$, and vertical y, a function of x and time t, its kinetic and potential energies are integrals that are derived from local data, the *Lagrange density* $\mathcal{L}$. Denote $y' = \frac{\partial y}{\partial x}$ and $\dot{y} = \frac{\partial y}{\partial t}$. Then

$$\mathcal{L}(y', \dot{y}) = \frac{1}{2}(\mu_0 y'^2 - T_0 \dot{y}^2),$$

where μ_0 is the string's mass density and T_0 the horizontal tension, both constant. With this structure, the Lagrangian $L = T - U$ at time t is

$$L(t) = \int_{x=0}^{x=a} \mathcal{L}$$

while the action is

$$S = \int_{t=t_0}^{t=t_1} L(t) = \int_{t=t_0}^{t=t_1} \int_{x=0}^{x=a} \mathcal{L}.$$

The variational calculus (similar to the calculation for point particles) yields a wave equation

$$0 = \mu_0 \frac{\partial^2 y}{\partial t^2} - T_0 \frac{\partial^2 y}{\partial x^2}$$

and two possible solutions for the boundary values of the string:

1. the *Neumann condition* $y'(t, 0) = y'(t, a) = 0$,
2. the *Dirichlet condition* $\dot{y}(t, 0) = \dot{y}(t, a) = 0$.

The first condition is what one gets when the boundary positions are free to move. The second condition fixes the y boundary positions over time, and this one is remarkable: It can be shown that the momentum of the string is conserved for the Neumann condition, but not necessarily for the Dirichlet condition. In this situation momentum can be exchanged with the points where the string is attached. This leads to the idea that such a string is connected to extended physical objects (one at $x = 0$, one at $x = a$) that may absorb some of the string's momentum. Accordingly, such objects are called *Dirichlet-branes* or *D-branes*.

While point particles, as they move in space-time, generate a *world-line*, strings create a *world-sheet*, see Figure K.1, that can be an open sheet (left in that figure) or a closed sheet, i.e., the trace of a closed string.

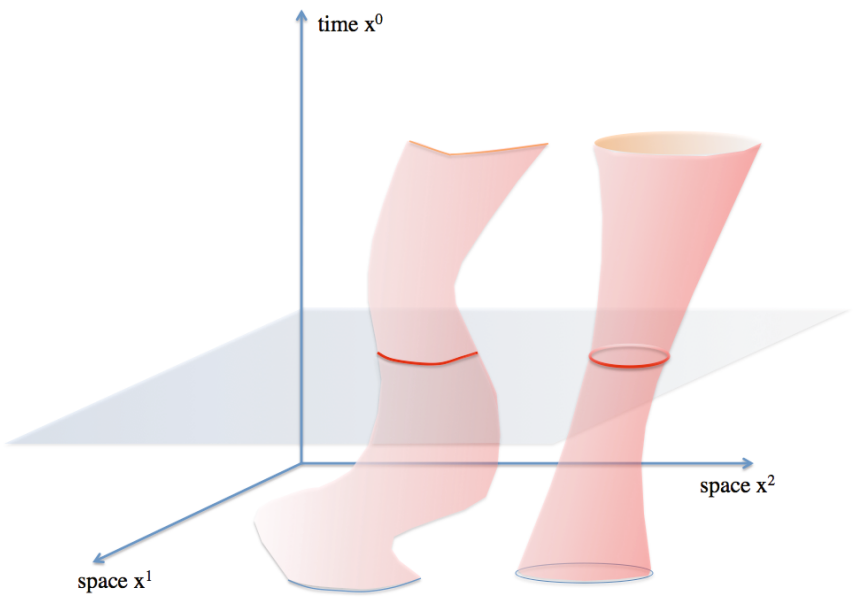

Fig. K.1. The world-sheet of a string, for open strings to the left, for closed strings (loops) to the right. The strings at a given time are the intersections of the world-sheet with the horizontal constant time hyperplane.

For the general relativistic theory of strings, the Lagrange action is defined using the surface of the world-sheet in terms of the Minkowski metric. The important fact here is to ensure that this surface is defined intrinsically, i.e., independent of the parametrization of the world-sheet. More precisely, the world-sheet is represented as a mapping $x : [a, b] \times [c, d] \to \mathbb{R}^4$ from a parameter rectangle $[a, b] \times [c, d]$ with arguments (ξ_1, ξ_2) (that could be normalized to I^2) to the physical space-time with the Minkowski metric. Using the classical scalar product[2] of two vectors, and the dot product in the Minkowski metric, the surface of the image of x is

$$A = \int \sqrt{\left(\frac{\partial x}{\partial \xi_1} \cdot \frac{\partial x}{\partial \xi_2}\right)^2 - \left(\frac{\partial x}{\partial \xi_1} \cdot \frac{\partial x}{\partial \xi_1}\right)\left(\frac{\partial x}{\partial \xi_2} \cdot \frac{\partial x}{\partial \xi_2}\right)}$$

This surface is independent of the parametrization. It can be expressed in a more compact way that also makes parametrization invariance more evident, using the matrix

$$g = \begin{pmatrix} \frac{\partial x}{\partial \xi_1} \cdot \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_1} \cdot \frac{\partial x}{\partial \xi_2} \\ \frac{\partial x}{\partial \xi_2} \cdot \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} \cdot \frac{\partial x}{\partial \xi_2} \end{pmatrix}$$

and then, we have

$$A = \int \sqrt{-det(g)}$$

which is transformed according to classical determinant rules and therefore is invariant under a change of parameters. Then the Lagrangian density is defined by

$$\mathcal{L}(\dot{x}, x') = -\frac{T_0}{c}\sqrt{-det(g)}$$

where $\dot{x}$ is the partial derivative with respect to the first parameter and x' is the partial derivative with respect to the second parameter. Reparametrizing the parameters to time τ and a length parameter σ, we get the *Nambu-Goto action* as

$$S = \int_{\tau_0}^{\tau_1} L = \int_{\tau_0}^{\tau_1} \int_{\sigma_0}^{\sigma_1} \mathcal{L},$$

where the Lagrangian is $L = \int_{\sigma_0}^{\sigma_1} \mathcal{L}$. The equations of motion, again deduced from Hamilton's principle, are as follows. Let

$$\mathcal{P}_\mu^\tau = \frac{\partial \mathcal{L}}{\dot{x}_\mu}, \mathcal{P}_\mu^\sigma = \frac{\partial \mathcal{L}}{x'_\mu}.$$

Then we have

$$\frac{\mathcal{P}_\mu^\tau}{\partial \tau} + \frac{\mathcal{P}_\mu^\sigma}{\partial \sigma} = 0.$$

for all μ.

K.3 Duality and Supersymmetry

In string theory, *strong-weak duality*, S-duality for short, is a duality that in particular exchanges strings and D-branes. Duality in theoretical physics is exemplified for Maxwell's equations for electric field E, magnetic field B, light speed constant c, and in absence of charge and current:

$$\nabla \cdot E = 0, \ \nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t}$$
$$\nabla \cdot B = 0, \ \nabla \times E = \frac{1}{c}\frac{\partial B}{\partial t}.$$

[2] The negative sign is due to the Minkowski metric's specificity.

The exchange of variables $(E, B) \mapsto (-B, E)$ leaves Maxwell's equations invariant, i.e., electromagnetism is *self-dual*. It can be shown that type IIB string theory with coupling constant g is dual to the same theory with coupling constant $\frac{1}{g}$, same for type I string theory with coupling constant g, which is dual to SO(32) theory with coupling constant $\frac{1}{g}$. The inversion of the coupling constant, turning a strong coupling into a weak one, and vice versa, is the reason for the name "S-duality". This type of relationship among different string theories is one reason for Edward Witten's 1995 suggestion that the five known string theories might be special cases of a single limit string theory, coined M-theory, "M" for Matrix, Mystery, Magic [1136].

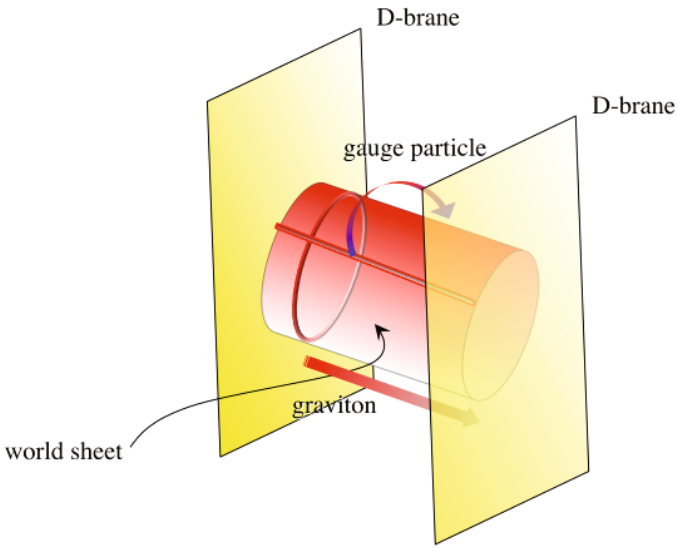

Fig. K.2. S-duality exchanging particles and D-branes.

Such a duality can exchange strings and D-branes, as shown in Figure K.2. Here a first interpretation is that an open gauge particle string, connecting two D-branes, by S-duality is interpreted as the exchange of a closed graviton string between two D-branes, this duality sending the fermonic gauge particle into a bosonic graviton, and reinterpreting D-branes as particles that exchange gravitational force. *Fermions* are those particles we usually associate with matter, e.g. electrons, protons, neutrons, quarks, whereas *bosons* are particles that transfer specific types of force, such as photons for the electromagnetic force, two W-bosons and one Z-boson for the weak force, gluons for the strong force, and (the still hypothetical) gravitons for gravitation; the Higgs boson (discovered in 2012) is involved in the unified electroweak force that unites electromagnetic and weak force types.

In theoretical physics, a theory that describes a symmetry between bosonic and fermionic particles is called supersymmetric. In string theory this approach is called a *superstring theory*. For this reason the above S-duality relates to superstring theory (but is of course not the same).

K.4 Quantum Mechanics

This part of the physical appendix refers to two books on mathematical methods in Quantum Mechanics (QM): Gerhard Teschl's well-written book [1034], and (much less) Leon A. Takhtajan's rather sketchy but also inspiring book [1033].

K.4.1 Banach and Hilbert Spaces

The mathematics of QM resides on Banach spaces and in particular Hilbert spaces. Banach spaces, i.e., complete topological $\mathbb{C}$-vector spaces with norms $\| \ \|$, were defined in Appendix Section I.1.2.

Example 140 A standard example is the space $\mathcal{C}^0[a,b]$ of continuous functions $f : [a,b] \to \mathbb{C}$ defined on the real closed interval $[a,b]$, the norm being

$$\|f\|_\infty = max_{x\in[a,b]} f(x).$$

Example 141 This Banach space $l^1(\mathbb{N})$ is the subspace of $\mathbb{C}^\mathbb{N}$ of all sequences $a. = (a_j)_j$ such that the norm

$$\|a.\|_1 = \sum_j |a_j|$$

is finite.

Example 142 The space $l^\infty(\mathbb{N})$ of all sequences in $\mathbb{C}^\mathbb{N}$ with finite norm

$$\|a.\|_\infty = sup_j |a_j|$$

is a Banach space.

Definition 217 *A* Schauder basis *of a normed $\mathbb{C}$-vector space X is a sequence $u. = (u_n)_{n\in\mathbb{N}}$ of linearly independent vectors such that every x is a limit $x = \lim_{n\to\infty} \sum_{m\leqslant n} c_n.u_n$. The space X is* separable *iff it contains a countable dense subset.*

Lemma 118 *A normed vector space is separable iff it has a Schauder basis.*

Theorem 103 (Weierstrass) *The subset of polynomial functions of $\mathcal{C}^0[a,b]$ is dense.*

Theorem 104 *$\mathcal{C}^0[a,b]$ with the maximum norm is a separable Banach space.*

While $\mathcal{C}^0[a,b]$ is separable, $l^\infty(\mathbb{N})$ is not.

A Hilbert space is a complex vector space H together with a *sesquilinear* form (usually called *scalar product*) $\langle , \rangle : H \times H \to \mathbb{C}$ which by definition means:

1. $\langle x, y\rangle$ it is linear in the right argument and conjugate linear in the left argument, i.e., $\langle x, \lambda y\rangle = \langle x, y\rangle\lambda$ while $\langle \lambda x, y\rangle = \overline{\lambda}\langle x, y\rangle$,
2. it is *positive definite*, i.e., $\langle x, x\rangle \geqslant 0$ and $\langle x, x\rangle = 0$ iff $x = 0$,
3. we have *symmetry*: $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

A *Hilbert space* is a Banach space whose norm is defined by the sesquilinear form via $\|x\| = \sqrt{\langle x, x\rangle}$. A vector space with an inner product is called an *inner product space.*

Example 143 The finite dimensional space $\mathbb{C}^n$ is a Hilbert space with the usual scalar product

$$\langle x, y\rangle = \sum_j^n \overline{x_j} y_j.$$

Example 144 The space $l^2(\mathbb{N})$ of sequences $a. \in \mathbb{C}^\mathbb{N}$ with $\sum_j |a_j|^2 < \infty$ is a separable Hilbert space with scalar product

$$\langle x, y\rangle = \sum_j^\infty \overline{x_j} y_j.$$

In an inner product space H, a vector with $\|x\| = 1$ is called *normalized* or *unit* vector. If $\langle x, y\rangle = 0$ one writes $x \perp y$ and calls them *orthogonal*, and if x is a multiple of y or vice versa, they are called *parallel*.

Proposition 138 (Cauchy-Schwarz-Bunjakowski) *For two vectors x, y in an inner product space we have*

$$|\langle x, y\rangle| \leqslant \|x\|\|y\|$$

with equality iff x and y are parallel.

The relationship between norms and scalar products is described by

Proposition 139 (Jordan-von Neumann) *A norm is associate with a scalar product iff the* parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + \|y\|^2$$

holds. In that case the scalar product is given by

$$\langle x, y\rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x - iy\|^2 - i\|x + iy\|^2).$$

Example 145 On $\mathcal{C}^0[a, b]$, there is a scalar product

$$\langle x, y\rangle = \int_a^b \overline{x(t)}y(t),$$

defining the inner product space $\mathcal{L}^2_{cont}([a, b])$.

If we are given two norms $\|\|_1, \|\|_2$ on one space, $\|\|_2$ is called *stronger than* $\|\|_1$ iff there is a positive real constant s such that $||x\|_1 \leqslant s\|x\|_2$. Then since $\|x\| \leqslant \sqrt{|b - a|}\|x\|_\infty$, the maximum norm is stronger than the $\mathcal{L}^2_{cont}$ norm.

Lemma 119 *If, for a normed space X, $\|\|_2$ is stronger than $\|\|_1$, than a Cauchy sequence for $\|\|_2$ is also a Cauchy sequence for $\|\|_1$, and if a set is dense in $(X, \|\|_2)$ it is also dense in $(X, \|\|_1)$.*

Therefore $\mathcal{L}^2_{cont}$ is separable, but not complete.

Proposition 140 *For a finite dimensional space, all norms are equivalent, i.e., mutually stronger. In particular, they define the same complete topology.*

Remark 36 *Normed spaces that are not complete can be turned into complete vector spaces by the standard process from calculus: Take the space of all Cauchy sequences and divide it by the subspace of zero convergent sequences.*

K.4.1.1 Bounded Operators

Definition 218 *A* linear operator *A between two normed spaces X, Y is a linear map $A : D(A) \to Y$, where $D(A)$, the* domain of *A isa dense linear subspace of X. We set $Ker(A)$ and $Rank(A)$ to be the kernel and image of the map $A : D(A) \to Y$. The operator A is* bounded *iff the* operator norm

$$\|A\| = sup_{\|x\|_X==1}\|A(x)\|_Y$$

is finite. The set of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$; we write $\mathcal{L}(X, X) = \mathcal{L}(X)$.

Theorem 105 *The space $\mathcal{L}(X, Y)$ together with the operator norm is a normed space that is Banach if Y is so. An operator A is bounded iff it is continuous.*

Here is the extension theorem:

Theorem 106 (Bounded Linear Transformation Theorem) *For $A \in \mathcal{L}(X,Y)$, if Y is Banach space, then there is a unique continuous extension of the domain $D(A)$ to all of X which has the same norm as A.*

Therefore it is no restriction to assume that an operator is defined on all of X.

The special space $X^* = \mathcal{L}(X,\mathbb{C})$ is called the *dual space of* X, and an operator in X^* is called a *bounded linear functional*.

The space $\mathcal{L}(X)$ has also the structure of a (unitary) $\mathbb{C}$-algebra defined by composition of operators. Moreover, one has $\|AB\| \leqslant \|A\|\|B\|$. A Banach space with such an algebra structure is called a *Banach algebra*.

K.4.1.2 Lebesque Integration

Definition 219 *A set $\Sigma \subset 2^X$ (X any set) is a* σ-algebra of X *if*

1. *$X \in \Sigma$,*
2. *Σ is closed under countable unions,*
3. *Σ is closed under complements.*

Therefore $\varnothing \in \Sigma$ and Σ is closed under countable intersections. Since the intersection of any non-empty family of σ-algebras of X is a σ-algebra, one can define the σ-algebra $\Sigma(S)$ generated by a subset $S \in 2^X$, which is the intersection of all σ-algebras containing S. Elements of Σ are called *measurable*.

Example 146 If X is a topological space, the *Borel σ-algebra* is the σ-algebra generated by the open sets. Its elements are called *Borel sets*. For $X = \mathbb{R}^n$, the Borel σ-algebra is denoted by $\mathcal{B}^n$, for $n = 1$ simply by $\mathcal{B}$.

Definition 220 *A* measure μ *is a map $\mu : \Sigma \to [0,\infty]$ on a σ-algebra Σ such that*

1. *$\mu(\varnothing) = 0$,*
2. *σ-addtitivity: $\mu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j)$ if The A_j are disjoint from each other.*
3. *μ is called* σ-finite *if there is a countable cover of X by measurable sets of Σ whose elements have finite measure. It is called* finite *if $\mu(X)$ is finite.*
4. *The pair (Σ,μ) is called a* measure space *(observe that X is determined by Σ).*
5. *A measure on the Borel σ-algebra is called a* Borel measure *if the measure of any quasi-compact set is finite.*

Given a σ-algebra Σ on X, a function $f : X \to \mathbb{R}^n$ is called *measurable* if $f^{-1}(B) \in \Sigma$ for every Borel set $B \in \mathcal{B}^n$. Measurable functions defined on the Borel σ-algebra of a topological space X are called *Borel functions*.

Lemma 120 *A function $f : X \to \mathbb{R}^n$ is measurable iff each projection $f_i : X \to \mathbb{R}$ is so.*

Lemma 121 *Borel functions are continuous. If $f : X \to Y$ with $Y \subset \mathbb{R}^n$ and $g : Y \to \mathbb{R}^m$ are Borel, then so is $g \circ f$. Let X be a topologic space and Σ its Borel σ-algebra. If $f, g : X \to \mathbb{R}$ are measurable functions, then so is $f + g$ and $f \cdot g$.*

Definition 221 *A measurable function $s : X \to \mathbb{R}$ is called* simple *if its image is finite. If its values are $\alpha_1, \ldots a_p$, using characteristic functions, we have $s = \sum_{i=1}^{p} \alpha_i \chi(A_i)$, $A_i = s^{-1}(\alpha_i) \in \Sigma$.*

Definition 222 *With the above notations, for a non-negative simple function s and a measurable set A, we define its integral by*

$$\int_A s d\mu = \sum_{i=1}^{p} \alpha_i \mu(A_i \cap A)$$

with the convention $0 \cdot \infty = 0$.

Sorite 18 *With the above notations, the integral has these basic properties:*

(*i*) $\int_A sd\mu = \int_X \chi(A)sd\mu$.
(*ii*) $\int_{\bigcup_{i=1}^{\infty} A_j} sd\mu = \sum_j \int_{A_j} sd\mu$ *for pairwise disjoint* A_j.
(*iii*) $\int_A \alpha sd\mu = \alpha \int_A sd\mu$ *for* $\alpha \geqslant 0$.
(*iv*) $\int_A (s+t)d\mu = \int_A sd\mu + \int_A td\mu$.
(*v*) $A \subset B$ *implies* $\int_A sd\mu \leqslant \int_B sd\mu$.
(*vi*) $s \leqslant t$ *implies* $\int_A sd\mu \leqslant \int_A td\mu$.

Definition 223 *IF* f *is any positive function on* X*, one defines*

$$\int_A f d\mu = sup_{s \leqslant f} \int_A sd\mu,$$

the supremum being taken over all simple functions $s \leqslant f$*. If both, the positive part* f^+ *and the negative part* f^- *of a measurable function* f *(defined by* $f = f^+ - f^-$*) are such that their integrals are finite, we define*

$$\int_A f d\mu = \int_A f^+ d\mu - \int_A f^- d\mu$$

and call f integrable*. The set of integrable functions is denoted by* $\mathcal{L}^1(X, d\mu)$*.*

Then Sorite 18 holds for integrable functions. Many standard theorems for Riemann integrals are also valid for Lebegue integrals, e.g. change of variables, Fubini, but see [1034, A.4] for examples.

K.4.1.3 Lebesgue L^p Spaces

Given a σ-finite measure space (Σ, μ), we denote for $p \geqslant 1$ by $\mathcal{L}^p(X, d\mu)$ the $\mathbb{C}$-linear space of $\mathbb{C}$-valued measurable functions with finite norm-like expression

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p}$$

In order to construct a Banach space from this data, one needs a result concerning vanishing of the norm:

Lemma 122 *Let* f *her measurable, then*

$$\int_X |f|^p d\mu = 0$$

iff $f(x) = 0$ *almost everywhere with respect to* μ*, i.e., if the null set of* f *has measure zero.*

Denote by $\mathcal{N}(X, d\mu)$ the set of measurable functions which are zero almost everywhere on X. This is a $\mathbb{C}$-linear subspace of $\mathcal{L}^p(X, d\mu)$, and we consider the quotient

$$L^p(X, d\mu) = \mathcal{L}^p(X, d\mu)/\mathcal{N}(X, d\mu)$$

which we also write as $L^P(X)$ if $X \subset \mathbb{R}^n$. Obviously, we now have a well-defined norm $\|f\|_p$ on equivalence classes of functions.

Theorem 107 $L^p(X, d\mu)$ *is a Banach space. If* X *has a countable basis (second countable) and* μ *is a regular Borel measure, then* $L^p(X, d\mu)$ *is separable.*

Regular Borel measures are defined in [1034, p.261].

K.4.2 Geometry on Hilbert Spaces

The phase spaces in QM are always Hilbert spaces, and their geometry is essential. We therefore discuss geometry on Hilbert spaces.

Example 147 The typical Hilbert space is $\mathcal{H} = L^2(X, d\mu)$ with the scalar product

$$\langle f, g\rangle = \int_X \overline{f(x)} g(x) d\mu$$

that induces the norm $\|\|_2$ discussed in Section K.4.1.3.

We already introduced unit or normalized vectors $\psi \in \mathcal{H}$, parallel vectors, and orthogonality in Section K.4.1. Given a unit vector ϕ, one defines two associated vectors to ψ, namely (1) the *ϕ-projection of ψ*: $\psi_\parallel = \langle \phi, \psi\rangle \phi$, and (2) the *associated orthogonal projection of ψ*: $\psi_\perp = \psi - \psi_\parallel$, whence $\psi = \psi_\parallel + \psi_\perp$.

Definition 224 *An orthonormal family of vectors is a family $(\phi_i)_i$ such that for all j, $\langle \phi_i, \phi_j\rangle = \delta_{ij}$, in particular, all are units.*

Lemma 123 *Given an orthonormal family $(\phi_1 \dots \phi_n)$, for a vector $\psi \in \mathcal{H}$, we set $\psi_\parallel = \sum_i \psi_{\parallel,\phi_i}$ and $\psi_\perp = \psi - \psi_\parallel$. Then $\psi_\parallel \perp \psi_\perp$ and $\phi_i \perp \psi_\perp$. We have* Bessel's inequality

$$\|\psi\|^2 \geqslant \sum_{i=1}^{n} |\langle \phi_i, \psi\rangle|^2,$$

equality holding iff $\psi \in \mathbb{C}\langle \phi_1 \dots \phi_n\rangle$. We have this formula for norms:

$$\|\psi\|^2 = \sum_i |\langle \phi_i, \psi\rangle|^2 + \|\psi_\perp\|^2.$$

The orthogonal projection is a well-known extremal solution as follows: If $\psi^* \in \mathbb{C}\langle \phi_1 \dots \phi_n\rangle$, then $\|\psi - \psi^*\| \geqslant \|\psi_\perp\|$, equality holding iff $\psi^* = \psi_\parallel$.

Bessel's inequality helps generalize these results to general orthonormal families:

Lemma 124 *Given an orthonormal family $(\phi_i)_i$, for a vector $\psi \in \mathcal{H}$, we set $\psi = \psi_\parallel + \psi_\perp$ with $\psi_\parallel = \sum_i \langle \phi_i, \psi\rangle \phi_i$, and we have $\psi_\parallel \perp \psi_\perp$. Also $\phi_i \perp \psi_\perp$. This implies (generalizing the same formula in Lemma 123)*

$$\|\psi\|^2 = \sum_i |\langle \phi_i, \psi\rangle|^2 + \|\psi_\perp\|^2.$$

Further, if $\psi^* \in \mathbb{C}\langle (\phi_i)_i\rangle$, then $\|\psi - \psi^*\| \geqslant \|\psi_\perp\|$, equality holding iff $\psi^* = \psi_\parallel$.

Definition 225 *Let $(\phi_i)_i$ be an orthonormal family of vectors in $\mathcal{H}$. Then they are an* orthonormal basis of *$\mathcal{H}$ if anyone of the following equivalent conditions holds.*

(i) $(\phi_i)_i$ *is a maximal orthonormal family.*
(ii) For all $\psi \in \mathcal{H}$, we have $\psi = \sum_i \langle \phi_i, \psi\rangle \phi_i$.
(iii) For all $\psi \in \mathcal{H}$, we have $\|\psi\|^2 = \sum_i |\langle \phi_i, \psi\rangle|^2$.
(iv) If $\langle \phi_i, \psi\rangle = 0$ for all i, then $\psi = 0$.

Example 148 A classical example is the orthornormal basis $\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{ins}$, $n \in \mathbb{Z}$ of $\mathcal{H} = L^2(0, 2\pi)$. The orthonormal series is the ordinary Fourier series.

Theorem 108 *A Hilbert space $\mathcal{H}$ is separable iff it has a countable orthonormal basis (then all orthonormal bases are countable[3]).*

Definition 226 *A linear isomorphism $f : \mathcal{H}_1 \to \mathcal{H}_2$ is called* unitary *iff it conserves the scalar products.*

Theorem 109 *Every infinite-dimensional separable Hilbert Space $\mathcal{H}$ is unitary isomorphic to $l^2(\mathbb{N})$. An isomorphism can be defined as follows. Take an orthonormal basis $(\phi_i)_i$ of $\mathcal{H}$ and define $f : \mathcal{H} \tilde{\to} l^2(\mathbb{N})$ by $f(\psi) = (\langle \phi_i, \psi \rangle)_i$*

If $M \subset \mathcal{H}$ is a subset, its *orthogonal complement* is $M^\perp = \{x | x \perp M\}$. It is a closed linear subspace. In particular, $0^\perp = \mathcal{H}$ by Definition 225.

Theorem 110 *Let M be a closed linear subspace of Hilbert space $\mathcal{H}$. Then we have a direct decomposition*

$$\mathcal{H} = M \oplus M^\perp$$

which means that every vector ψ has the unique representation $\psi = \psi_\parallel + \psi_\perp, \psi_\parallel \in M, \psi_\perp \in M^\perp$. Also $M^{\perp\perp} = M$.

Evidently, we then have the *orthogonal projection* $p_M : \mathcal{H} \to \mathcal{H} : \psi \mapsto \psi_\parallel$ which is idempotent and verifies $\langle p_M \psi, \phi \rangle = \langle \psi_\parallel, \phi_\parallel \rangle = \langle \psi, p_M \phi \rangle$. The other projection $p_{M^\perp} : \mathcal{H} \to \mathcal{H} : \psi \mapsto \psi_\perp = \psi - p_M \psi$ is also idempotent and verifies $\langle p_{M^\perp} \psi, \phi \rangle = \langle \psi_\perp, \phi_\perp \rangle = \langle \psi, p_{M^\perp} \phi \rangle$. If $P_M \neq 0$ then $\|P_M\| = 1$.

Lemma 125 (Riesz Lemma) *For every bounded linear functional $l : \mathcal{H} \to \mathbb{C}$ there is a unique vector $\lambda \in \mathcal{H}$ with*

$$l(\psi) = \langle \psi, \lambda \rangle$$

This means that the map $\mathcal{H} \to \mathcal{H}^* : \lambda \mapsto \langle ?, \lambda \rangle$ is a linear isomorphism.

Corollary 41 *Suppose that s is a bounded sesquilinear form, i.e., $|s(\psi, \phi)| \leqslant C\|\psi\|\|\phi\|$, then there is a unique bounded operator B such that $s(\psi, \phi) = \langle B\psi, \phi \rangle$, moreover $\|B\| \leqslant C$. In particular, if $A \in \mathcal{L}(\mathcal{H})$, then there is the* adjoint operator *$A^* \in \mathcal{L}(\mathcal{H})$ such that $\langle \phi, A^* \psi \rangle = \langle A\phi, \psi \rangle$.*

Lemma 126 *Let $A, B \in \mathcal{L}(\mathcal{H})$. Then*

(i) (conjugate linearity) $(A + B)^* = A^* + B^*$ and $(\lambda A)^* = \overline{\lambda} A^*$,
(ii) (involution) $A^{**} = A$,
(iii) (antimultiplicative) $(AB)^* = B^* A^*$,
(iv) $\|A\| = \|A^*\|$ and $\|A\|^2 = \|A^* A\| = \|AA^*\|$.

Definition 227 *If $\mathcal{H}_1, \langle , \rangle_1$ and $\mathcal{H}_2, \langle , \rangle_2$ are two Hilbert spaces, their* orthogonal direct sum *$\mathcal{H}_1 \oplus^\perp \mathcal{H}_2$ is the algebraic direct sum with the orthogonal sum of their scalar products*

$$\langle (\phi_1, \phi_2), (\psi_1, \psi_2) \rangle_{1 \oplus 2} = \langle \phi_1, \psi_1 \rangle_1 + \langle \phi_2, \psi_2 \rangle_2.$$

The direct sum $\bigoplus_i^\perp \mathcal{H}_i$ of a countable family $(\mathcal{H}_i)_i$ of Hilbert spaces is the subspace of the algebraic direct sum consisting of the vectors (ψ_i) with $\sum_i \|\psi_i\|^2 < \infty$ and the orthogonal sum of the scalar products of the factors.

Example 149 *We have $\bigoplus_i \mathbb{C} = l^2(\mathbb{N})$.*

[3] This is automatic for linear bases, but here we deal with Schauder bases.

Definition 228 *If $\mathcal{H}_1, \langle,\rangle_1$ and $\mathcal{H}_2, \langle,\rangle_2$ are two Hilbert spaces, their* (orthogonal) tensor product $\mathcal{H}_1 \otimes^\perp \mathcal{H}_2$ *is the completion of the algebraic tensor product for the norm that is induced by the scalar product*

$$\langle \phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2 \rangle_{1\otimes 2} = \langle \phi_1, \psi_1 \rangle_1 \cdot \langle \phi_2, \psi_2 \rangle_2.$$

If $(\phi_i^1)_i), (\phi_i^2)_i)$ are orthonormal bases for $\mathcal{H}_1, \mathcal{H}_2$, respectively, then the tensor family $(\phi_i^1 \otimes \phi_j^2)_{i,j}$ is an orthonormal basis of $\mathcal{H}_1 \otimes^\perp \mathcal{H}_2$.

Example 150 We have $\mathcal{H} \otimes \mathbb{C}^n = \mathcal{H}^n$.

K.4.2.1 The C^*-Algebra of Bounded Linear Operators

The properties (i) to (iv) in Lemma 126 turn $\mathcal{L}(\mathcal{H})$ into a *C^*-algebra*. A subalgebra that is closed under the * involution is called a *$*$-subalgebra*. An *ideal* in such an algebra is an ordinary two-sided ideal. It is a 8-ideal if it is stable under the adjunction map. An element a in a C^*-algebra is called *normal* if $aa^* = a^*a$ (it commutes with its adjoint), *self-adjoint* if $a = a^*$, *unitary* if it is normal and $aa^* = Id$, *an (orthogonal) projection* if it is self-adjoint and idempotent, and *positive* if $a = bb^*$ for some b.

Lemma 127 *If $A \in \mathcal{L}(\mathcal{H})$, then A is normal iff $\|A\psi\| = \|A^*\psi\|$ for all ψ.*

Lemma 128 *A transformation $T : \mathcal{H} \to \mathcal{H}$ is unitary iff $T^{-1} = T^*$.*

K.4.3 Axioms for Quantum Mechanics

Here is the axiomatic setup of QM. To define a quantum system we are given a Hilbert space $\mathcal{H}$ which is called *phase space* or *configuration space*. The *states* of the quantum system are the unit vectors $\psi \in \mathcal{H}$ (the elements of the unit sphere $S^1 \subset \mathcal{H}$). An *observable* is a self-adjoint linear operator A on $\mathcal{H}$.

The expectation of an observable A in state ψ is defined by the quadratic form value, i.e.,

$$E_\psi(A) = \langle \psi, A\psi \rangle.$$

Example 151 Here is an example that illustrates this definition. It is the single particle in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$, i.e., three-space and time. It is described by a $\mathbb{C}$-valued *wave function* $\psi : \mathbb{R}^4 \to \mathbb{C}$. The square absolute value $|\psi(x,t)|^2 = \overline{\psi(x,t)}\psi(x,t)$ defines the *probability density* of the particle at (x,t). This quantity must be defined such that the space integral of density at any time t is one (the particle must for sure be somewhere at any time):

$$\int_{\mathbb{R}^3} |\psi(x,t)|^2 = 1.$$

Here, the observable corresponds to the location x of the particle. This location is however not a point but an extended spatial area $\Omega \subset \mathbb{R}^3$, such as the interior of a detector. We may then want to calculate the probability for the particle to live in Ω at time t. This probability is the integral of the probability density over Ω:

$$E_\psi(\chi(\Omega)) = \int_\Omega |\psi(x,t)|^2.$$

But this expression can be rephrased by

$$\int_\Omega |\psi(x,t)|^2 = \int_{\mathbb{R}^3} \overline{\psi(x,t)}\chi(\Omega)\psi(x,t)$$

which is the above quadratic form expression for the scalar product $\langle \phi, \psi \rangle = \int_{\mathbb{R}^3} \overline{\phi}\psi$ and the operator $A(\psi) = \chi(\Omega)\psi$.

The time evolution of a system is given by a *one parameter unitary group*, which means the following: It is a one-parameter family $(U(t))_{t\geqslant 0}$ of linear unitary operators $U(t) : \mathcal{H} \to \mathcal{H}$ such that $U(t)\circ U(s) = U(t+s)$, in particular $U(0) = Id$, and such that the group is *strongly continuous*, i.e., $\lim_{t\to t_0} U(t)\psi = U(t_0)\psi$.

The group is given a *Hamilton operator*

$$H\psi = \lim_{t\to 0} \frac{i}{t}(U(t)\psi - \psi),$$

and the domain $\mathcal{D}(H)$ is the subset of $\mathcal{H}$ where $H\psi$ exists. If $\psi \in \mathcal{D}(H)$, then we have the *Schrödinger equation*

$$i\frac{d}{dt}\psi(t) = H\psi(t).$$

These structures define the axioms of QM: (1) The configuration space is a separable Hilbert space $\mathcal{H}$, the system states are the unit vectors on the unit sphere in $\mathcal{H}$. (2) An observable corresponds to a linear selfadjoint operator. (3) The expectation value for a measurement of an observable A in state ψ is the real valued quadratic form $\langle A\psi, \psi\rangle$. (4) the time evolution of the system, starting at state ψ, is given by a strongly continuous one-parameter unitary group $(U(t))$. The Hamilton operator $H\psi$ corresponds to the *energy of the system at state* ψ.

K.4.3.1 Resolvents and Spectra

Eigenvectors and -values play an important role in QM. For infinite-dimensional spaces, these concepts needs to be defined in a generalized way compared to the finite-dimensional situation.

Definition 229 *The* resolvent set *of a selfadjoint operator A on $\mathcal{H}$ is defined as $\rho(A) = \{z \in \mathbb{C} | (A-z)^{-1} \in \mathcal{L}(\mathcal{H})\}$. The function $R_A : \rho(A) \to \mathcal{L}(\mathcal{H}) : z \mapsto (A-z)^{-1}$ is called the* resolvent of A.

The complement $\sigma(A) = \mathbb{C} - \rho(A)$ is called the *spectrum of A*. If $z \in \sigma(A)$, this means that there are non-zero vectors ψ with $A\psi = z\psi$, z is called an *eigenvalue* and ψ an *eigenvector for z*.

Lemma 129 *For a selfadjoint operator all eigenvalues are real, and eigenvectors to different eigenvalues are orthogonal.*

Theorem 111 *Let U be unitary, then $\sigma(U)$ consists of unit vectors (elements of the unit circles $S^1 \subset \mathbb{C}$). Eigenvectors for different eigenvalues are orthogonal.*

K.4.4 The Spectral Theorem

The spectral theorem relates to the fact that a measurement of an observable A results in an eigenvalue of A (which is a real number according to Lemma 129).

K.4.4.1 Projection-valued Measures

Definition 230 *Let $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{R}$. A* projection-valued measure *is a function $P : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ to the set of orthogonal projections on $\mathcal{H}$ (recall from Section K.4.2.1: such a projection p is idempotent and self-adjoint) with the following conditions:*

(i) $P(\mathbb{R}) = Id$.
(ii) If $\Omega = \bigcup_n \Omega_n$ *with* $\Omega_n \cap \Omega_m = \varnothing$ *for* $m \neq n$, *then* $\sum_n P(\Omega_n)\psi = P(\Omega)\psi$ *for all* ψ.

Example 152 —*em Let $\mathcal{H} = \mathbb{C}^n$ and let $A \in GL(n)$ be a symmetric matrix. If $\lambda_1, \ldots \lambda_m$ are its different eigenvalues, denote by $P_1, \ldots P_m$ the orthogonal projections onto the corresponding eigenspaces. Then*

$$P_A(\Omega) = \sum_{j|\lambda_j \in \Omega} P_j$$

is a projection valued measure.

Sorite 19 *If P is a projection-valued measure, then*

(*i*) $P(\varnothing) = 0$,
(*ii*) $P(\mathbb{R} - \Omega) = Id - P(\Omega)$,
(*iii*) $P(\Omega_1 \cup \Omega_2) + P(\Omega_1 \cap \Omega_2) = P(\Omega_1) + P(\Omega_2)$,
(*iv*) $P(\Omega_1)P(\Omega_2) = P(\Omega 1 \cap \Omega_2)$,
(*v*) *if $\Omega_1 \subset \Omega_2$, then $P(\Omega_1) \leqslant P(\Omega_2)$.*

In particular, one can define a *resolution of the identity* for every projection-valued measure P, namely $P(\lambda) = P(]-\infty, \lambda])$, $\lambda \in \mathbb{R}$. This enables us to extend the concept of Lebegue integration to projection-valued measures. For a simple function $f = \sum_{i=1}^n \alpha_i \chi(\Omega_j)$ with the usual $\Omega_j = f^{-1}(\alpha_j)$, one sets

$$P(f) = \int_{\mathbb{R}} f(\lambda) dP(\lambda) = \sum_{i=1}^n \alpha_i P(\Omega_j),$$

which defines an operator instead of a complex number. For two state vectors ϕ, ψ, one then has a complex Borel measure $\mu_{\phi,\psi}(\Omega) = \langle \phi, P(\Omega) \rangle$, and this relates our operator-valued integration to the complex-valued one by the formula

$$\langle \phi, P(f)\psi \rangle = \int_{\mathbb{R}} f(\lambda) d\mu_{\phi,\psi}(\lambda).$$

The definition of this operator-valued integral can be extended to the Banach space of bounded Borel functions $B(\mathbb{R})$ and yields a function

$$P : B(\mathbb{R}) \to \mathcal{L}(\mathcal{H}) : f \mapsto \int_{\mathbb{R}} f(\lambda) dP(\lambda)$$

and the above equation relating operator-valued integration to the complex-valued one remains valid.

Theorem 112 *Given a projection-valued measure P on $\mathcal{H}$, the transformation*

$$P : B(\mathbb{R}) \to \mathcal{L}(\mathcal{H}) : f \mapsto \int_{\mathbb{R}} f(\lambda) dP(\lambda)$$

is a C algebra homomorphism with norm one such that*

$$\langle P(g)\phi, P(f)\psi \rangle = \int_{\mathbb{R}} g^*(\lambda) f(\lambda) d\mu_{\phi,\psi}(\lambda).$$

The following *Spectral Theorem* (John von Neumann) decomposes any self-adjoint operator A into its "projection spectrum".

Theorem 113 (Spectral Theorem) *For every self-adjoint operator A there is a unique projection-valued measure P_A such that*

$$A = \int_{\mathbb{R}} \lambda P_A(\lambda).$$

This enables in particular the calculation of A's spectrum:

Proposition 141 *The spectrum of operator A is described by*

$$\sigma(A) = \{\lambda \in \mathbb{R} | P_A(\ ]\lambda - \varepsilon, \lambda + \varepsilon[\ ) \neq 0 \text{ for all } \varepsilon > 0\}.$$

Part XXIII

Appendix: Tables

L

Euler's Gradus Function

This table lists the rational numbers x/y with Euler's gradus suavitatis $\Gamma(x/y) \leqslant 10$, see also [161].

Γ	Intervals
2	1/2
3	1/3, 1/4
4	1/6,2/3,1/8
5	1/5,1/9,1/12,3/4,1/16
6	1/10,2/5,1/18,2/9,1/24,3/8,1/32
7	1/7,1/15,3/5,1/20,4/5,1/27,1/36,4/9,1/48,3/16,1/64
8	1/14,2/7,1/30,2/15,3/10,5/6,1/40,5/8,1/54,2/27,1/72,8/9,1/96,3/32,1/128
9	1/21,3/7,1/25,1/28,4/7,1/45,5/9,1/60,3/20,4/15,5/12,1/80,5/16,1/81,1/108, 4/27,1/144,9/16,1/192,3/64,1/256
10	1/42,2/21,3/14,6/7,1/50,2/25,1/56,7/8,1/90,2/45,5/18,9/10,1/120,3/40,5/24, 8/15,1/160,5/32,1/162,2/81,1/216,8/27,1/288,9/32,1/384,3/128,1/512

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M

Just and Well-Tempered Tuning

This table lists the just coordinates of the just tuning intervals (with respect to c, second tone in first column) according to Vogel [1089], see Section 7.2.1.4, together with the value in Cents, and the deviation in % from the tempered tuning with 100, 200, 300, etc. Cents.

Tone name	Frequency ratio	Octave coord.	Fifth coord.	Third coord.	Pitch (Ct)	% deviation
c	1	0	0	0	0	0
$d_\flat$	16/15	4	-1	-1	111.73	+11.73
d	9/8	-3	2	0	203.91	+1.96
$e_\flat$	6/5	1	1	-1	315.65	+5.22
e	5/4	-2	0	1	386.31	-3.42
f	4/3	2	-1	0	498.05	-0.39
$f_\sharp$	45/32	-5	2	1	590.22	-1.63
g	3/2	-1	1	0	701.96	+0.28
$a_\flat$	8/5	3	0	-1	813.69	+1.71
a	5/3	0	-1	1	884.36	-1.74
$b_\flat$	16/9	4	-2	0	996.09	-0.39
b	15/8	-3	1	1	1088.27	-1.07

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N

Chord and Third Chain Classes

N.1 Chord Classes

This section contains the list of all isomorphism classes of zero-addressed chords in $PiMod_{12}$. The meanings of the column items are explained in Section 11.3.7; here we give a short definition.

- *Class Nr.* is the number of the isomorphism class, numbers with extension ".1" indicate the class number for classification under symmetries from $\mathbb{Z}$ (no fifth or fourth transformations). Autocomplementary classes have a star after the number.
- *Representative* of Nr. without hat is the number's representative in full circles, the one with hat is the complementary chord.
- *Group of symmetries* is $Sym(Nr.)$. To keep notation readable, we use the notation with linear factor to the left.
- *Conj. Class* denotes the conjugacy class symbol of $Sym(Nr.)$ and refers to the numbering $1, 2, \ldots 19$ from [799].
- *Card. End. Cl.* $Nr.|\widehat{Nr.}$ is the pair of numbers of conjugacy classes of endomorphisms in *Nr.* and in its complement $\widehat{Nr.}$, respectively.

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Chord Classes

Class Nr.	*Representative* $Nr. = \bullet,\ \widehat{Nr.} = \circ$	*Group of Symmetries*	*Conj. Class*	♯ *End.* $Nr.\vert\widehat{Nr.}$
1	●●●●●●●●●●●●	$\overrightarrow{GL}(\mathbb{Z}_{12})$	19	28|28
		One/Eleven Element		
2	●○○○○○○○○○○○	$\mathbb{Z}_{12}^{\times}$	8	1|31
		Two/Ten Elements		
3	●●○○○○○○○○○○	$\langle -1e^{-1} \rangle$	3	3|23
3.1	●○○○○●○○○○○			
4	●○●○○○○○○○○○	$\{1, 7, -1e^{-2}, 5e^{-2}\}$	8	3|25
5	●○○●○○○○○○○○	$\{1, 5, 7e^{-3}, -1e^{-3}\}$	8	3|19
6	●○○○●○○○○○○○	$\{1, 7, 5e^{8}, -1e^{8}\}$	8	3|31
7	●○○○○○●○○○○○	$\mathbb{Z}_{12}^{\times} \ltimes e^{6\mathbb{Z}_{12}}$	13	3|28
		Three/Nine Elements		
8	●●●○○○○○○○○○	$\langle -1e^{-2} \rangle$	2	4|14
8.1	●○●○○○○○●○○○			
9	●●○●○○○○○○○○	$\{1\}$	1	4|30
9.1	●○○○○○○○○○○○			
10	●●○○●○○○○○○○	$\{1\}$	1	8|36
10.1	●○○●○○○○●○○○			
11	●●○○○○●○○○○○	$\langle 5 \rangle$	4	4|20
12	●●○○○○○●○○○○	$\langle 7e^{6} \rangle$	6	5|29
13	●○●○●○○○○○○○	$\{1, 7, -1e^{8}, 5e^{8}\}$	8	4|18
14	●○●○○○●○○○○○	$\langle 7 \rangle$	6	8|31
15	●○○●○○●○○○○○	$\{1, 5, -1e^{6}, 7e^{6}\}$	8	5|32
16	●○○○●○○○●○○○	$\mathbb{Z}_{12}^{\times} \ltimes e^{4\mathbb{Z}_{12}}$	15	4|20
		Four/Eight Elements		
17	●●●●○○○○○○○○	$\langle -1e^{-3} \rangle$	3	4|8
17.1	●○●○○●○●○○○○			
18	●●●○●○○○○○○○	$\{1\}$	1	5|19
18.1	●○●○●○○●○○○○			
19	●●●○○●○○○○○○	$\{1\}$	1	5|19
19.1	●●○●○○○○●○○○			
20	●●●○○○●○○○○○	$\{1\}$	1	7|23
20.1	●●○○○●○●○○○○			
21	●●●○○○○●○○○○	$\{1, 7, -1e^{-2}, 5e^{-2}\}$	9	7|9
22	●●○●●○○○○○○○○	$\langle -1e^{-4} \rangle$	2	6|20
22.1	●○●○○●○○○●○○			
23	●●○●○●○○○○○○	$\langle 5 \rangle$	4	5|13
24	●●○●○○●○○○○○	$\langle 7e^{6} \rangle$	6	6|17

Chord Classes—Continued				
Class Nr.	*Representative* $Nr. = \bullet,\ \widehat{Nr.} = \circ$	*Group of Symmetries*	*Conj. Class*	♯ *End.* $Nr.\vert\widehat{Nr.}$
25	●●○●○○○●○○○○	$\{1\}$	1	0|3
25.1	●●○●○●○○○○○○			
26	●●○●○○○○○○●○○	$\{1\}$	1	12|31
26.1	●○●○○●○○●○○○			
27	●●○●○○○○○○○●○	$\{1, 7e^{-3}, 5e^{2}, -1e^{-1}\}$	11	5|13
28	●●○○●●○○○○○○	$\langle -1e^{7} \rangle$	3	6|14
28.1	●●○○○●○○●○○○			
29	●●○○●○○●○○○○	$\langle 7 \rangle$	6	10|23
30	●●○○●○○○●○○○	$\langle 5e^{4} \rangle$	4	11|23
31	●●○○●○○○○●○○	$\{1, -1e^{-1}, 5e^{-4}, 7e^{3}\}$	10	9|19
32	●●○○○●●○○○○○	$\{1, 5, -1e^{6}, 7e^{6}\}$	8	7|15
33	●●○○○○●●○○○○	$\{1, 7, -1e^{-1}, 5e^{-1}, e^{6}, 7e^{6}, 5e^{5}, -1e^{5}\}$	14	7|14
34	●○●○●○●○○○○○	$\{1, 7, -1e^{6}, 5e^{6}\}$	9	6|17
35	●○●○●○○○●○○○	$\{1, 7, -1e^{4}, 5e^{4}\}$	8	11|19
36	●○●○○○●○●○○○	$\{1, 7, -1e^{-2}, 5e^{-2}, e^{6}, 7e^{6}, 5e^{4}, -1e^{4}\}$	13	9|28
37	●○○●○○●○○●○○	$\mathbb{Z}_{12}^{\times} \ltimes e^{3\mathbb{Z}_{12}}$	17	7|21
Five/Seven Elements				
38	●●●●●○○○○○○○	$\langle -1e^{-4} \rangle$	2	5|7
38.1	●○●○●○○●○●○○			
39	●●●●○●○○○○○○	$\{1\}$	1	6|10
39.1	●●○●○●○○○○●○			
40	●●●●○○●○○○○○	$\{1\}$	1	8|12
40.1	●●○●○○●○●○○○			
41	●●●●○○○●○○○○	$\{1\}$	1	8|12
41.1	●●●○○●○●○○○○			
42	●●●○●●○○○○○○	$\{1\}$	1	6|16
42.1	●●○●○●○○●○○○			
43	●●●○●○●○○○○○	$\{1\}$	1	8|20
43.1	●●○●○●○●○○○○			
44	●●●○●○○●○○○○	$\langle 7 \rangle$	6	7|9
45	●●●○●○○○●○○○	$\{1\}$	1	16|22
45.1	●●○○●○●○●○○○			
46	●●●○●○○○○●○○	$\langle 5e^{-4} \rangle$	4	5|12
47	●●●○●○○○○○●○	$\langle -1e^{-2} \rangle$	2	8|14
47.1	●○●○●○●○○●○○			

Chord Classes—Continued				
Class Nr.	*Representative* $Nr. = \bullet,\ \widehat{Nr.} = \circ$	*Group of Symmetries*	*Conj. Class*	♯ *End.* $Nr.\vert\widehat{Nr.}$
48	●●●○○●●○○○○○	$\{1\}$	1	8|18
48.1	●●○●○○○●●○○○			
49	●●●○○●○○●○○○	$\{1\}$	1	10|18
49.1	●●○●○○○○●●○○			
50	●●●○○●○○○●○○	$\langle -1e^{-2} \rangle$	2	9|13
50.1	●●○●●○○○●○○○			
51	●●●○○○●●○○○○	$\langle 7 \rangle$	6	9|11
52	●●●○○○●○●○○○	$\{1, -1e^{-2}, 7e^{6}, 5e^{4}\}$	8	7|17
53	●●○●●○●○○○○○	$\{1\}$	1	10|20
53.1	●●○●○○●○○○●○			
54	●●○●●○○●○○○○	$\{1\}$	1	14|26
54.1	●●○○●○●○○●○○			
55	●●○●○●●○○○○○	$\{1, 5, -1e^{6}, 7e^{6}\}$	8	8|8
56	●●○●○●○○○●○○	$\langle 5 \rangle$	4	16|16
57	●●○●○○●●○○○○	$\langle 7e^{6} \rangle$	6	12|16
58	●●○●○○●○○●○○	$\langle 7e^{6} \rangle$	6	18|23
59	●●○●○○○●○●○○	$\langle 7 \rangle$	6	13|29
60	●●○○●●○○●○○○	$\langle 5 \rangle$	4	11|19
61	●●○○●○○●●○○○	$\{1, 7, -1e^{4}, 5e^{4}\}$	8	14|14
62	●○●○●○●○●○○○	$\{1, 7, -1e^{4}, 7e^{4}\}$	8	11|19
Six/Six Elements				
63*	●●●●●●○○○○○○	$\langle -1e^{-5} \rangle$	3	5|5
63.1*	●●○●○●○○●○●○			
64*	●●●●●○●○○○○○	$\{1\}$	1	9|9
64.1*	●●○●○●○●○○●○			
65	●●●●●○○●○○○○	$\{1\}$	1	9|9
65.1	●●●○●○○●○●○○			
66	●●●●●○○○●○○○	$\langle -1e^{-4} \rangle$	2	12|6
66.1	●●●○○●○●○●○○			
67*	●●●●○●○●○○○○	$\langle 5e^{-2} \rangle$	5	6|6
68*	●●●●○●○○●○○○	$\{1\}$	1	9|9
69	●●●●○●○○○●○○	$\{1\}$	1	15|11
69.1	●●○●●○●○●○○○			
70*	●●●●○●○○○○●○	$\{1, 5, -1e^{-3}, 7e^{-3}\}$	10	6|6
71*	●●●●○○●●○○○○	$\{1\}$	1	11|11
71.1*	●●●○○●○●●○○○			
72	●●●●○○●○●○○○	$\langle 7e^{6} \rangle$	6	8|10

Chord Classes—Continued

Class Nr.	*Representative* $Nr. = \bullet$, $\widehat{Nr.} = \circ$	*Group of Symmetries*	*Conj. Class*	♯ *End.* $Nr.\|\widehat{Nr.}$
73	●●●●○○●○○●○○	$\langle -1e^{-3} \rangle$	3	13|9
73.1	●●○●○○●○●●○○			
74*	●●●●○○○●●○○○	$\langle -1e^{-3} \rangle$	3	7|7
75*	●●●○●●○○●○○○	$\{1\}$	1	17|17
75.1*	●●○○●●○○●○●○			
76*	●●●○●●○○○●○○	$\langle 5e^{-5} \rangle$	4	10|10
77*	●●●○●○●○●○○○	$\langle 5e^{4} \rangle$	4	14|14
78*	●●●○●○●○○○●○	$\{1\}$	1	23|23
78.1*	●●○●○●○●○●○○			
79	●●●○●○○●●○○○	$\langle 7 \rangle$	6	18|10
80	●●●○●○○●○○●○	$\{1, 7, 5e^{-2}, -1e^{-2}\}$	9	15|11
81*	●●●○●○○○●●○○	$\langle 5e^{-4} \rangle$	4	11|11
82*	●●●○○●●○○●○○	$\{1\}$	1	17|17
82.1*	●●○●●○○●●○○○			
83*	●●●○○○●●●○○○	$\{1, -1e^{-2}, 5e^{-2}, 7\} \ltimes e^{6\mathbb{Z}_{12}}$	13	12|12
84*	●●○●●○●○○●○○	$\langle 7e^{3} \rangle$	7	14|14
85	●●○●●○○●○●○○	$\{1, -1e^{-4}, 5e^{-4}, 7\}$	8	15|23
86*	●●○●○○●●○●○○	$\langle 7 \rangle \ltimes e^{6\mathbb{Z}_{12}}$	12	20|20
87*	●●○○●●○○●●○○	$\{1, e^4, e^8, 5, 5e^4, 5e^8, -1e^{-1}, -1e^3, -1e^7, 7e^{-1}, 7e^3, 7e^7\}$	16	12|12
88*	●○●○●○●○●○●○	$\mathbb{Z}_{12}^{\times} \ltimes e^{\mathbb{Z}_{12}}$	18	12|12

N.2 Third Chain Classes

The following list of third chain translation classes shows the class number in the first column, where equivalence (∼) means that the same *pc* set is generated. The second column shows the pitch classes in the order of appearance along the third chain. The third column shows the third chain, the fourth column shows the chord class of the *pc* set, and the fifth column shows lead-sheet symbols as systematically derived in Section 25.2.1.

Third Chains				
Chain Nr. ∼ *equiv.*	*Pitch Classes from 0*	*Third Chain*	*Chord Class*	*Lead-Sheet Symbols*
Two Pitch Classes				
1	0,3	3	5	trd
2	0,4	4	6	Trd
Three Pitch Classes				
3	0,3,6	33	15	$C0, Cm5$-
4	0,3,7	34	10.1	Cm
5	0,4,7	43	10.1	C
6	0,4,8	44	16	$C+, C5+$
Four Pitch Classes				
7	0,3,6,9	333	37	$C07$-
8	0,3,6,10	334	26.1	$C07$
9	0,3,7,10	343	22.1	$Cm7$
10	0,3,7,11	344	30	$Cm7+$
11	0,4,7,10	433	26.1	$C7$
12	0,4,7,11	434	28.1	$C7+$
13	0,4,8,11	443	30	$C+7+$
Five Pitch Classes				
14	0,3,6,9,1	3334	58	$C07$-/9-
15	0,3,6,10,1	3343	53.1	$C09$-
16	0,3,6,10,2	3344	56	$C09$
17	0,3,7,10,1	3433	53.1	$Cm9$-
18	0,3,7,10,2	3434	42.1	$Cm9$
19	0,3,7,11,2	3443	59	$Cm7+/9, Cmmaj7/9$
20	0,4,7,10,1	4333	58	$C9$-
21	0,4,7,10,2	4334	47.1	$C9$
22	0,4,7,11,2	4343	42.1	$C7+/9, Cmaj7/9$
23	0,4,7,11,3	4344	60	$C7+/9+$
24	0,4,8,11,2	4433	56	$C+7+/9$
25	0,4,8,11,3	4434	60	$C+7+/9+$
Six Pitch Classes				
26	0,3,6,9,1,4	33343	84*	$C07$-/9-/11-
27	0,3,6,9,1,5	33344	85^	$C07$-/9-/11
28	0,3,6,10,1,4	33433	79	$C09$-/11-
29	0,3,6,10,1,5	33434	65.1^	$C09$-/11

Third Chains—Continued				
Chain Nr. ∼ *equiv.*	*Pitch Classes from 0*	*Third Chain*	*Chord Class*	*Lead-Sheet Symbols*
30	0,3,6,10,2,5	33443	69.1	$C011$
31	0,3,7,10,1,4	34333	84*	$Cm9\text{-}/11\text{-}$
32	0,3,7,10,1,5	34334	64.1*	$Cm9\text{-}/11$
33	0,3,7,10,2,5	34343	63.1*	$Cm11$
34	0,3,7,10,2,6	34344	75.1*	$Cm11+$
35	0,3,7,11,2,5	34433	69.1	$Cm7+/11$
36	0,3,7,11,2,6	34434	82*	$Cm7+/11+$
37	0,4,7,10,1,5	43334	73.1	$C9\text{-}/11$
38	0,4,7,10,2,5	43343	64.1*	$C11$
39	0,4,7,10,2,6	43344	78.1*	$C11+$
40	0,4,7,11,2,5	43433	65.1^	$C7+/11$
41	0,4,7,11,2,6	43434	66.1^	$C7+/11+$
42	0,4,7,11,3,6	43443	82*	$C7+/9+/11+$
43	0,4,8,11,2,5	44333	85^	$C+7+/11$
44	0,4,8,11,2,6	44334	78.1*	$C+7+/11+$
45	0,4,8,11,3,6	44343	75.1*	$C+7+/9+/11+$
46	0,4,8,11,3,7	44344	87*	$C+7+/9+/(11)/13\text{-}$
Seven Pitch Classes				
47	0,3,6,9,1,4,7	333433	58^	$C07\text{-}/9\text{-}/11\text{-}/13\text{-}$
48	0,3,6,9,1,4,8	333434	54.1^	$C07\text{-}/9\text{-}/11\text{-}/13$
49	0,3,6,9,1,5,8	333443	54.1^	$C07\text{-}/9\text{-}/13$
50	0,3,6,10,1,4,7	334333	58^	$C09\text{-}/11\text{-}/13\text{-}$
51	0,3,6,10,1,4,8	334334	47.1	$C09\text{-}/11\text{-}/13$
52	0,3,6,10,1,5,8	334343	38.1	$C09\text{-}/13$
53	0,3,6,10,1,5,9	334344	54.1^	$C09\text{-}/13+$
54	0,3,6,10,2,5,8	334433	47.1^	$C013$
55	0,3,6,10,2,5,9	334434	54.1^	$C013+$
56	0,3,7,10,1,4,8	343334	54.1^	$Cm9\text{-}/11\text{-}/13$
57	0,3,7,10,1,5,8	343343	38.1^	$Cm9\text{-}/13$
58	0,3,7,10,1,5,9	343344	47.1^	$Cm9\text{-}/13+$
59	0,3,7,10,2,5,8	343433	38.1^	$Cm13$
60	0,3,7,10,2,5,9	343434	38.1^	$Cm13+$
61	0,3,7,10,2,6,9	343443	54.1^	$Cm11+/13+$
62	0,3,7,11,2,5,8	344333	54.1^	$Cm7+/13$
63	0,3,7,11,2,5,9	344334	47.1^	$Cm7+/13+$
64	0,3,7,11,2,6,9	344343	54.1^	$Cm7+/11+/13+$
65	0,3,7,11,2,6,10	344344	60^	$Cm7+/11+/(13)/15\text{-}$
66	0,4,7,10,1,5,8	433343	54.1^	$C9\text{-}/13$
67	0,4,7,10,1,5,9	433344	54.1^	$C9\text{-}/13+$
68	0,4,7,10,2,5,8	433433	47.1^	$C13$
69	0,4,7,10,2,5,9	433434	38.1^	$C13+$

Third Chains—Continued				
Chain Nr. ∼ *equiv.*	*Pitch Classes from 0*	*Third Chain*	*Chord Class*	*Lead-Sheet Symbols*
70	0,4,7,10,2,6,9	433443	47.1^	C11+/13+
71	0,4,7,11,2,5,8	434333	54.1^	C7+/13
72	0,4,7,11,2,5,9	434334	38.1^	C7+/13+
73	0,4,7,11,2,6,9	434343	38.1^	C7+/11+/13+
74	0,4,7,11,2,6,10	434344	45.1^	C7+/11+/(13)/15-
75	0,4,7,11,3,6,9	434433	54.1^	C7+/9+/11+/13+
76	0,4,7,11,3,6,10	434434	55^	C7+/9+/11+/(13)/15-
77	0,4,8,11,2,5,9	443334	54.1^	C+7+/13+
78	0,4,8,11,2,6,9	443343	47.1^	C+7+/11+/13+
79	0,4,8,11,2,6,10	443344	62^	C+7+/11+/(13)/15-
80	0,4,8,11,3,6,9	443433	54.1^	C+7+/9+/11+/13+
81	0,4,8,11,3,6,10	443434	45.1^	C+7+/9+/11+/(13)/15-
82	0,4,8,11,3,7,10	443443	60^	C+7+/9+/(11)/13-/15-
Eight Pitch Classes				
83	0,3,6,9,1,4,7,10	3334333	37^	C07-/9-/11-/13-...
84	0,3,6,9,1,4,7,11	3334334	26.1^	C07-/9-/11-/13-...
85	0,3,6,9,1,4,8,11	3334343	22.1^	C07-/9-/11-/13...
86	0,3,6,9,1,5,8,11	3334433	26.1^	C07-/9-/13...
87	0,3,6,10,1,4,7,11	3343334	29^	C09-/11-/13-...
88	0,3,6,10,1,4,8,11	3343343	18.1^	C09-/11-/13...
89	0,3,6,10,1,5,8,11	3343433	17.1^	C09-/13...
90	0,3,6,10,2,5,8,11	3344333	26.1^	C013...
91	0,3,6,10,2,5,9,1	3344344	31^	C013+...
92	0,3,7,10,1,4,8,11	3433343	31^	Cm9-/11-/13...
93	0,3,7,10,1,5,8,11	3433433	18.1^	Cm9-/13...
94	0,3,7,10,2,5,8,11	3434333	22.1^	Cm13...
95	0,3,7,10,2,5,9,1	3434344	18.1^	Cm13+...
96	0,3,7,10,2,6,9,1	3434434	29^	Cm11+/13+...
97	0,3,7,11,2,5,9,1	3443344	34^	Cm7+/13+...
98	0,3,7,11,2,6,9,1	3443434	29^	Cm7+/11+/13+...
99	0,3,7,11,2,6,10,1	3443443	28^	Cm7+/11+/(13)/15-...
100	0,4,7,10,1,5,8,11	4333433	29^	C9-/13...
101	0,4,7,10,2,5,8,11	4334333	18.1^	C15
102	0,4,7,10,2,5,9,1	4334344	22.1^	C13+...
103	0,4,7,10,2,6,9,1	4334434	26.1^	C11+/13+...
104	0,4,7,11,2,5,9,1	4343344	18.1^	C7+/13+...
105	0,4,7,11,2,6,9,1	4343434	17.1^	C7+/11+/13+...
106	0,4,7,11,2,6,10,1	4343443	25.1^	C7+/11+/(13)/15-...
107 ∼ 84	0,4,7,11,3,6,9,1	4344334	26.1^	C7+/9+/11+/13+...
108 ∼ 87	0,4,7,11,3,6,10,1	4344343	29^	C7+/9+/11+/(13)/15-...
109	0,4,7,11,3,6,10,2	4344344	30^	C7+/9+/11+/(13)/15-...

Third Chains—Continued				
Chain Nr. ∼ *equiv.*	*Pitch Classes from 0*	*Third Chain*	*Chord Class*	*Lead-Sheet Symbols*
110	0,4,8,11,2,5,9,1	4433344	31^	C+7+/13+ . . .
111	0,4,8,11,2,6,9,1	4433434	18.1^	C+7+/11+/13+ . . .
112	0,4,8,11,2,6,10,1	4433443	34^	C+7+/11+/(13)/15- . . .
113 ∼ 85	0,4,8,11,3,6,9,1	4434334	22.1^	C+7+/9+/11+/13+ . . .
114 ∼ 88	0,4,8,11,3,6,10,1	4434343	18.1^	C+7+/9+/11+/(13)/15- . . .
115	0,4,8,11,3,6,10,2	4434344	35^	C+7+/9+/11+/(13)/15- . . .
116 ∼ 92	0,4,8,11,3,7,10,1	4434433	31^	C+7+/9+/(11)/13-/15- . . .
117	0,4,8,11,3,7,10,2	4434434	30^	C+7+/9+/(11)/13-/15- . . .
Nine Pitch Classes				
118	0,3,6,9,1,4,7,10,2	33343334	15^	C07-/9-/11-/13- . . .
119	0,3,6,9,1,4,7,11,2	33343343	9.1^	C07-/9-/11-/13- . . .
120	0,3,6,9,1,4,8,11,2	33343433	9.1^	C07-/9-/11-/13 . . .
121	0,3,6,9,1,5,8,11,2	33344333	15^	C07-/9-/13 . . .
122	0,3,6,10,1,4,7,11,2	33433343	10^	C09-/11-/13- . . .
123	0,3,6,10,1,4,8,11,2	33433433	13^	C09-/11-/13 . . .
124	0,3,6,10,1,5,8,11,2	33434333	9.1^	C09-/13 . . .
125	0,3,6,10,2,5,9,1,4	33443443	10^	C013+ . . .
126	0,3,7,10,1,4,8,11,2	34333433	10^	Cm9-/11-/13 . . .
127	0,3,7,10,1,5,8,11,2	34334333	9.1^	Cm9-/13 . . .
128	0,3,7,10,2,5,9,1,4	34343443	9.1^	Cm13+ . . .
129	0,3,7,10,2,6,9,1,4	34344343	15^	Cm11+/13+ . . .
130	0,3,7,11,2,5,9,1,4	34433443	13^	Cm7+/13+ . . .
131 ∼ 119	0,3,7,11,2,6,9,1,4	34434343	9.1^	Cm7+/11+/13+ . . .
132	0,3,7,11,2,6,9,1,5	34434344	14^	Cm7+/11+/13+ . . .
133 ∼ 122	0,3,7,11,2,6,10,1,4	34434433	10^	Cm7+/11+/(13)/15- . . .
134	0,3,7,11,2,6,10,1,5	34434434	11^	Cm7+/11+/(13)/15- . . .
135	0,4,7,10,1,5,8,11,2	43334333	15^	C9-/13 . . .
136	0,4,7,10,1,5,8,11,3	43334334	10.1^	C9-/13 . . .
137	0,4,7,10,2,5,8,11,3	43343334	10.1^	C17
138	0,4,7,10,2,6,9,1,5	43344344	10.1^	C11+/13+ . . .
139	0,4,7,11,2,6,9,1,5	43434344	8.1^	C7+/11+/13+ . . .
140	0,4,7,11,2,6,10,1,5	43434434	12^	C7+/11+/(13)/15- . . .
141	0,4,7,11,3,6,9,1,5	43443344	14^	C7+/9+/11+/13+ . . .
142	0,4,7,11,3,6,10,1,5	43443434	12^	C7+/9+/11+/(13)/15- . . .
143	0,4,7,11,3,6,10,2,5	43443443	11^	C7+/9+/11+/(13)/15- . . .
144	0,4,8,11,2,6,9,1,5	44334344	10.1^	C+7+/11+/13+ . . .
145	0,4,8,11,2,6,10,1,5	44334434	14^	C+7+/11+/(13)/15- . . .
146	0,4,8,11,3,6,9,1,5	44343344	10.1^	C+7+/9+/11+/13+ . . .
147	0,4,8,11,3,6,10,1,5	44343434	8.1^	C+7+/9+/11+/(13)/15- . . .
148	0,4,8,11,3,6,10,2,5	44343443	14^	C+7+/9+/11+/(13)/15- . . .
149 ∼ 136	0,4,8,11,3,7,10,1,5	44344334	10.1^	C+7+/9+/(11)/13-/15- . . .

Third Chains—Continued				
Chain Nr. ∼ *equiv.*	*Pitch Classes from 0*	*Third Chain*	*Chord Class*	*Lead-Sheet Symbols*
150 ∼ 137	0,4,8,11,3,7,10,2,5	44344343	10.1^	C+7+/9+/(11)/13-/15- . . .
151	0,4,8,11,3,7,10,2,6	44344344	16^	C+7+/9+/(11)/13-/15- . . .
Ten Pitch Classes				
152	0,3,6,9,1,4,7,10,2,5	333433343	5^	C07-/9-/11-/13- . . .
153	0,3,6,9,1,4,7,11,2,5	333433433	4^	C07-/9-/11-/13- . . .
154	0,3,6,9,1,4,8,11,2,5	333434333	5^	C07-/9-/11-/13 . . .
155	0,3,6,10,1,4,7,11,2,5	334333433	3^	C09-/11-/13- . . .
156	0,3,6,10,1,4,8,11,2,5	334334333	4^	C09-/11-/13 . . .
157 ∼ 152	0,3,6,10,2,5,9,1,4,7	334434433	5^	C013+ . . .
158	0,3,6,10,2,5,9,1,4,8	334434434	6^	C013+ . . .
159	0,3,7,10,1,4,8,11,2,5	343334333	5^	Cm9-/11-/13 . . .
160	0,3,7,10,1,4,8,11,2,6	343334334	6^	Cm9-/11-/13 . . .
161	0,3,7,10,1,5,8,11,2,6	343343334	3.1^	Cm9-/13 . . .
162	0,3,7,10,2,5,9,1,4,8	343434434	3.1^	Cm13+ . . .
163	0,3,7,10,2,6,9,1,4,8	343443434	7^	Cm11+/13+ . . .
164	0,3,7,11,2,5,9,1,4,8	344334434	6^	Cm7+/13+ . . .
165	0,3,7,11,2,6,9,1,4,8	344343434	3.1^	Cm7+/11+/13+ . . .
166	0,3,7,11,2,6,9,1,5,8	344343443	7^	Cm7+/11+/13+ . . .
167 ∼ 160	0,3,7,11,2,6,10,1,4,8	344344334	6^	Cm7+/11+/(13)/15- . . .
168 ∼ 161	0,3,7,11,2,6,10,1,5,8	344344343	3.1^	Cm7+/11+/(13)/15- . . .
169	0,3,7,11,2,6,10,1,5,9	344344344	6^	Cm7+/11+/(13)/15- . . .
170	0,4,7,10,1,5,8,11,2,6	433343334	7^	C9-/13 . . .
171	0,4,7,10,1,5,8,11,3,6	433343343	3.1^	C9-/13 . . .
172	0,4,7,10,2,5,8,11,3,6	433433343	6^	C19
173	0,4,7,10,2,6,9,1,5,8	433443443	6^	C11+/13+ . . .
174	0,4,7,11,2,6,9,1,5,8	434343443	3.1^	C7+/11+/13+ . . .
175 ∼ 170	0,4,7,11,2,6,10,1,5,8	434344343	7^	C7+/11+/(13)/15- . . .
176	0,4,7,11,2,6,10,1,5,9	434344344	3.1^	C7+/11+/(13)/15- . . .
177 ∼ 171	0,4,7,11,3,6,9,1,5,8	434433443	3.1^	C7+/9+/11+/13+ . . .
178	0,4,7,11,3,6,10,1,5,8	434434343	6^	C7+/9+/11+/(13)/15- . . .
179	0,4,7,11,3,6,10,1,5,9	434434344	7^	C7+/9+/11+/(13)/15- . . .
180 ∼ 172	0,4,7,11,3,6,10,2,5,8	434434433	6^	C7+/9+/11+/(13)/15- . . .
181	0,4,7,11,3,6,10,2,5,9	434434434	3.1^	C7+/9+/11+/(13)/15- . . .
182	0,4,8,11,2,6,10,1,5,9	443344344	6^	C+7+/11+/(13)/15- . . .
183	0,4,8,11,3,6,10,1,5,9	443434344	3.1^	C+7+/9+/11+/(13)/15- . . .
184	0,4,8,11,3,6,10,2,5,9	443434434	7^	C+7+/9+/(11)/13-/15- . . .
185	0,4,8,11,3,7,10,1,5,9	443443344	6^	C+7+/9+/(11)/13-/15- . . .
186	0,4,8,11,3,7,10,2,5,9	443443434	3.1^	C+7+/9+/(11)/13-/15- . . .
187	0,4,8,11,3,7,10,2,6,9	443443443	6^	C+7+/9+/(11)/13-/15- . . .
Eleven Pitch Classes				
188	0,3,6,9,1,4,7,10,2,5,8	3334333433	2^	C07-/9-/11-/13- . . .

Third Chains—Continued				
Chain Nr. ∼ *equiv.*	*Pitch Classes from 0*	*Third Chain*	*Chord Class*	*Lead-Sheet Symbols*
189	0,3,6,9,1,4,7,11,2,5,8	3334334333	2^	C07-/9-/11-/13- . . .
190	0,3,6,10,1,4,7,11,2,5,8	3343334333	2^	C09-/11-/13- . . .
191	0,3,6,10,1,4,7,11,2,5,9	3343334334	2^	C09-/11-/13- . . .
192	0,3,6,10,1,4,8,11,2,5,9	3343343334	2^	C09-/11-/13 . . .
193 ∼ 191	0,3,6,10,2,5,9,1,4,7,11	3344344334	2^	C013+ . . .
194 ∼ 192	0,3,6,10,2,5,9,1,4,8,11	3344344343	2^	C013+ . . .
195	0,3,7,10,1,4,8,11,2,5,9	3433343334	2^	Cm9-/11-/13 . . .
196	0,3,7,10,1,4,8,11,2,6,9	3433343343	2^	Cm9-/11-/13 . . .
197	0,3,7,10,1,5,8,11,2,6,9	3433433343	2^	Cm9-/13 . . .
198 ∼ 195	0,3,7,10,2,5,9,1,4,8,11	3434344343	2^	Cm13+ . . .
199	0,4,7,10,1,5,8,11,2,6,9	4333433343	2^	C9-/13 . . .
200	0,4,7,10,1,5,8,11,3,6,9	4333433433	2^	C9-/13 . . .
201	0,4,7,10,2,5,8,11,3,6,9	4334333433	2^	C21
202 ∼ 199	0,4,7,10,2,6,9,1,5,8,11	4334434433	2^	C11+/13+ . . .
203 ∼ 191	0,4,7,11,3,6,10,2,5,9,1	4344344344	2^	C7+/9+/11+/(13)/15- . . .
204 ∼ 192	0,4,8,11,3,6,10,2,5,9,1	4434344344	2^	C+7+/9+/(11)/13-/15- . . .
205 ∼ 195	0,4,8,11,3,7,10,2,5,9,1	4434434344	2^	C+7+/9+/(11)/13-/15- . . .
206 ∼ 196	0,4,8,11,3,7,10,2,6,9,1	4434434434	2^	C+7+/9+/(11)/13-/15- . . .
Twelve Pitch Classes				
207	0,3,6,9,1,4,7,10,2,5,8,11	33343334333	1^	C07-/9-/11-/13- . . .
208 ∼ 207	0,4,7,10,2,5,8,11,3,6,9,1	43343334334	1^	C23
209 ∼ 207	0,4,7,10,2,6,9,1,5,8,11,3	43344344334	1^	C11+/13+ . . .
210 ∼ 207	0,4,8,11,3,7,10,2,6,9,1,5	44344344344	1^	C+7+/9+/(11)/13-/15- . . .

O

Two, Three, and Four Tone Motif Classes

O.1 Two Tone Motifs in $OnPiMod_{12,12}$

$ClassNr.$	$Representative$
1	$(0,0),(0,1)$
2	$(0,0),(0,2)$
3	$(0,0),(0,3)$
4	$(0,0),(0,4)$
5	$(0,0),(0,6)$

O.2 Two Tone Motifs in $OnPiMod_{5,12}$

$ClassNr.$	$Representative$
1	$(0,0),(0,1)$
2	$(0,0),(0,2)$
3	$(0,0),(0,3)$
4	$(0,0),(0,4)$
5	$(0,0),(0,6)$
6	$(0,0),(1,0)$
7	$(0,0),(1,1)$
8	$(0,0),(1,2)$
9	$(0,0),(1,2)$
10	$(0,0),(1,4)$
11	$(0,0),(1,6)$

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O.3 Three Tone Motifs in $OnPiMod_{12,12}$

Refer to the discussion in Section 11.3.8 for the entries of this table. The order of these representatives is a historical one. After this table, the representatives are also visualized on a 12×12 square in list O.1.

Three-Element Motif Classes in $OnPiMod_{12,12}$				
Class Nr.	*Representative*	*Kernel*	*Class Weight*	*Volume*
1	$(0,0),(1,0),(2,0)$	$\mathbb{Z}.(1,2) \times \mathbb{Z}.(1,1)$	$(1,1,2)$	0
2	$(0,0),(1,0),(3,0)$	$\mathbb{Z}.(1,2) \times \mathbb{Z}.(0,1)$	$(1,2,3)$	0
3	$(0,0),(1,0),(4,0)$	$\mathbb{Z}.(1,0) \times \mathbb{Z}.(0,1)$	$(1,3,4)$	0
4	$(0,0),(1,0),(5,0)$	$\mathbb{Z}.(1,2) \times \mathbb{Z}.(1,1)$	$(1,1,4)$	0
5	$(0,0),(1,0),(6,0)$	$\mathbb{Z}.(1,2) \times \mathbb{Z}.(1,0)$	$(1,1,6)$	0
6	$(0,0),(2,0),(4,0)$	$(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times \mathbb{Z}.(1,1)$	$(2,2,4)$	0
7	$(0,0),(2,0),(6,0)$	$(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times \mathbb{Z}.(0,1)$	$(2,4,6)$	0
8	$(0,0),(3,0),(6,0)$	$\mathbb{Z}.(1,2) \times \mathbb{Z}_3^2$	$(3,3,6)$	0
9	$(0,0),(4,0),(8,0)$	$(\mathbb{Z}_4^2) \times \mathbb{Z}.(1,1)$	$(4,4,4)$	0
10	$(0,0),(1,0),(0,1)$	0×0	$(1,1,1)$	1
11	$(0,0),(2,0),(0,1)$	$\mathbb{Z}.(2,0) \times 0$	$(1,1,2)$	2
12	$(0,0),(3,0),(0,1)$	$0 \times \mathbb{Z}.(1,0)$	$(1,1,3)$	3
13	$(0,1),(0,2),(3,0)$	$0 \times \mathbb{Z}.(1,1)$	$(1,1,1)$	3
14	$(0,0),(0,1),(4,0)$	$\mathbb{Z}.(1,0) \times 0$	$(1,1,4)$	4
15	$(0,0),(1,2),(2,0)$	$\mathbb{Z}.(1,2) \times 0$	$(1,1,2)$	4
16	$(0,0),(2,0),(0,2)$	$2\mathbb{Z}_4^2 \times 0$	$(2,2,2)$	4
17	$(0,0),(6,0),(0,1)$	$\mathbb{Z}.(2,0) \times \mathbb{Z}.(1,0)$	$(1,1,6)$	6
18	$(0,0),(3,0),(0,2)$	$\mathbb{Z}.(2,0) \times \mathbb{Z}.(0,1)$	$(1,2,3)$	6
19	$(0,0),(0,2),(3,1)$	$\mathbb{Z}.(2,0) \times \mathbb{Z}.(1,1)$	$(1,1,2)$	6
20	$(0,0),(4,0),(0,2)$	$(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times 0$	$(2,2,4)$	4
21	$(0,0),(4,0),(0,4)$	$0 \times \mathbb{Z}_3^2$	$(3,3,3)$	3
22	$(0,0),(6,0),(0,2)$	$2\mathbb{Z}_4^2 \times \mathbb{Z}.(1,0)$	$(2,2,6)$	0
23	$(0,2),(0,4),(6,0)$	$2\mathbb{Z}_4^2 \times \mathbb{Z}.(1,1)$	$(2,2,2)$	0
24	$(0,0),(4,0),(0,4)$	$\mathbb{Z}_4^2 \times 0$	$(4,4,4)$	4
25	$(0,0),(6,0),(0,3)$	$\mathbb{Z}.(2,0) \times \mathbb{Z}_3^2$	$(3,3,6)$	6
26	$(0,0),(6,0),(0,6)$	$2\mathbb{Z}_4^2 \times \mathbb{Z}_3^2$	$(6,6,6)$	0

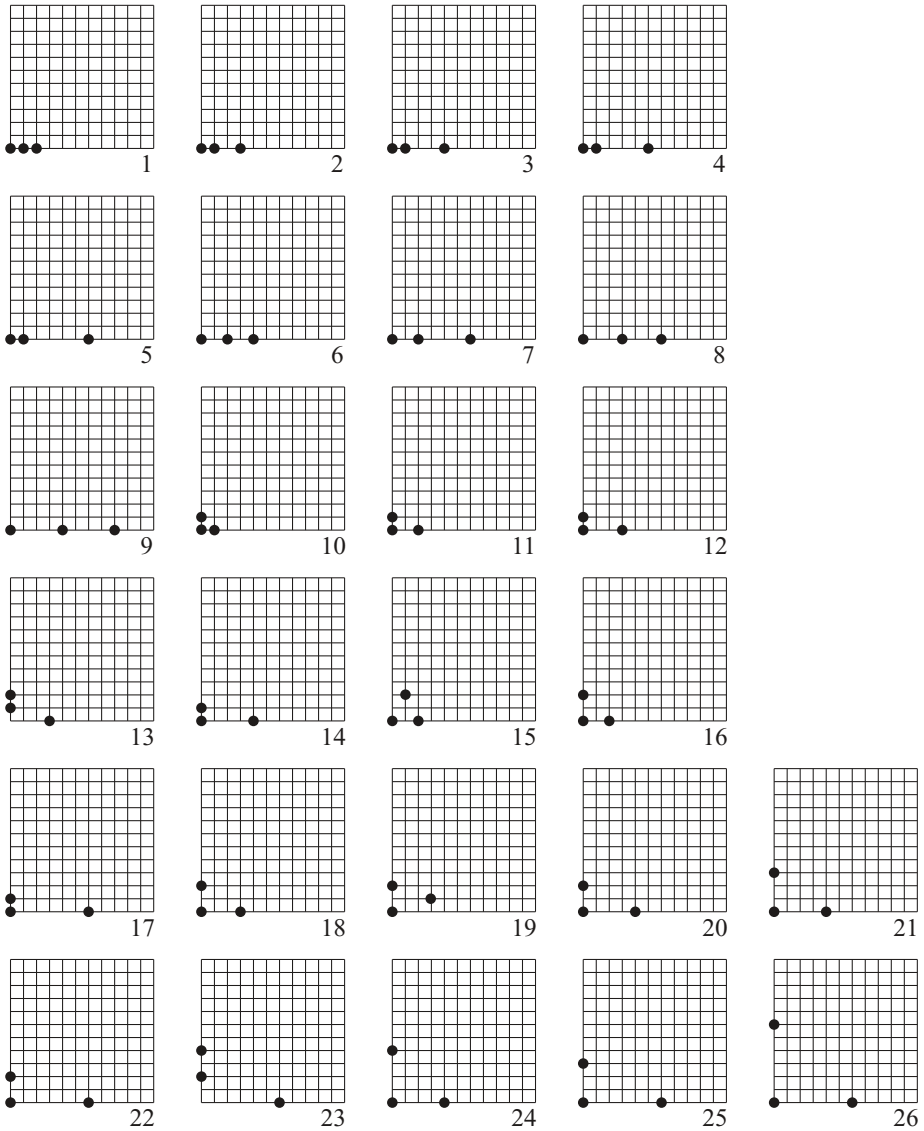

Fig. O.1. Representatives of the 26 isomorphism classes of three-element motives in $OnPiMod_{12,12}$.

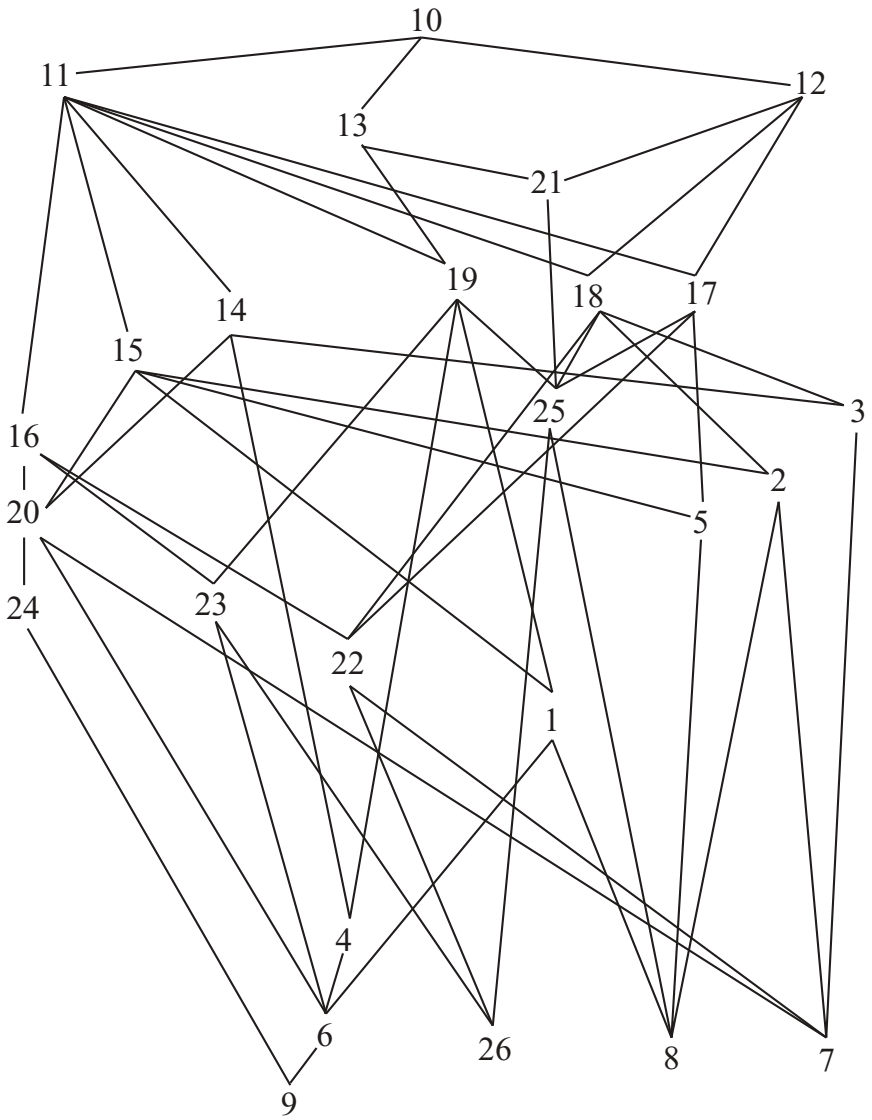

Fig. O.2. Hasse diagram of dominance/specialization among the 26 isomorphism classes of motives in $OnPiMod_{12,12}$.

O.4 Four Tone Motifs in $OnPiMod_{12,12}$

This list was calculated by Straub in [1017], refer to Section 11.3.8 for details. The list's numbering follows Straub's algorithm; * denotes classes which are not determined by volume and class weight.

Four-Element Motif Classes			
Class Nr.	*Representative*	*Class Weight*	*Volume*
0	(0,0),(0,1),(0,2),(0,7)	(1,1,5,5)	0
1	(0,0),(0,1),(0,2),(0,3)	(1,1,2,2)	0
2	(0,0),(0,1),(0,2),(0,6)	(1,4,5,7)	0
3	(0,0),(0,1),(0,2),(0,5)	(1,4,2,3)	0
4	(0,0),(0,1),(0,2),(0,4)	(1,2,3,6)	0
5	(0,0),(1,0),(0,5),(0,6)	(4,4,5,5)	0
6	(0,0),(0,1),(0,4),(0,5)	(4,4,3,3)	0
7	(0,0),(0,1),(0,3),(0,5)	(4,2,2,6)	0
8	(0,0),(0,1),(0,4),(0,8)	(4,3,3,9)	0
9	(0,0),(0,1),(0,6),(0,7)	(5,5,5,5)	0
10	(0,0),(0,1),(0,3),(0,6)	(5,2,2,8)	0
11	(0,0),(0,1),(0,3),(0,7)	(5,2,3,7)	0
12	(0,0),(0,1),(0,4),(0,7)	(5,3,3,8)	0
13	(0,0),(0,1),(0,3),(0,10)	(2,2,2,2)	0
14	(0,0),(0,1),(0,3),(0,4)	(2,2,3,3)	0
15	(0,0),(0,1),(0,3),(0,9)	(2,3,7,8)	0
16	(0,0),(0,1),(0,4),(0,9)	(3,3,3,3)	0
17	(0,0),(0,2),(6,0),(6,10)	(23,23,22,22)	0
18	(0,0),(0,2),(0,4),(6,0)	(23,6,22,7)	0
19	(0,0),(0,2),(0,4),(6,2)	(6,6,22,22)	0
20	(0,0),(0,2),(0,4),(0,6)	(6,6,7,7)	0
21	(0,0),(0,2),0,4),(0,8)	(6,7,7,9)	0
22	(0,0),(0,2),(6,0),(6,2)	(22,22,22,22)	0
23	(0,0),(0,2),(6,0),(6,6)	(22,22,22,26)	0
24	(0,0),(0,2),(0,6),(6,2)	(22,22,7,7)	0
25	(0,0),(0,2),(0,6),(6,0)	(22,7,7,26)	0
26	(0,0),(0,2),(0,6),(0,8)	(7,7,7,7)	0
27	(0,0),(0,3),(0,6),(0,9)	(8,8,8,8)	0
28	(0,0),(0,6),(6,0),(6,6)	(26,26,26,26)	0
29	(0,0),(0,1),(0,2),(1,0)	(1,10,10,11)	1
30	(0,0),(0,1),(0,5),(1,0)	(4,10,10,14)	1
31	(0,0),(0,1),(0,6),(1,0)	(5,10,10,17)	1
32	(0,0),(0,1),(0,3),(1,0)	(2,10,11,12)	1
33	(0,0),(0,1),(0,4),(1,0)	(3,10,12,14)	1
34*	(0,0),(0,1),(1,0),(1,5)	(10,10,10,10)	1

Four-Element Motif Classes—Continued			
Class Nr.	*Representative*	*Class Weight*	*Volume*
35*	(0,0),(0,1),(1,0),(7,7)	(10,10,10,10)	1
36*	(0,0),(0,1),(1,0),(1,1)	(10,10,10,10)	1
37*	(0,0),(0,1),(1,0),(3,5)	(10,10,10,13)	1
38*	(0,0),(0,1),(1,0),(3,11)	(10,10,10,13)	1
39*	(0,0),(0,1),(1,0),(1,2)	(10,10,11,11)	1
40*	(0,0),(0,1),(1,0),(5,10)	(10,10,11,11)	1
41*	(0,0),(0,1),(1,0),(4,10)	(10,10,11,15)	1
42*	(0,0),(0,1),(1,0),(2,4)	(10,10,11,15)	1
43	(0,0),(0,1),(1,0),(2,5)	(10,10,11,19)	1
44*	(0,0),(0,1),(1,0),(1,3)	(10,10,12,12)	1
45*	(0,0),(0,1),(1,0),(7,9)	(10,10,12,12)	1
46*	(0,0),(0,1),(1,0),(3,3)	(10,10,12,12)	1
47	(0,0),(0,1),(1,0),(6,8)	(10,10,15,19)	1
48*	(0,0),(0,1),(1,0),(1,4)	(10,10,14,14)	1
49*	(0,0),(0,1),(1,0),(4,4)	(10,10,14,14)	1
50	(0,0),(0,1),(1,0),(1,6)	(10,10,17,17)	1
51	(0,0),(0,1),(1,0),(2,2)	(10,11,11,13)	1
52	(0,0),(0,1),(1,0),(2,3)	(10,11,13,15)	1
53	(0,0),(0,1),(1,0),(6,9)	(10,11,12,18)	1
54	(0,0),(0,1),(1,0),(8,8)	(10,13,14,14)	1
55	(0,0),(0,1),(1,0),(3,4)	(10,12,15,18)	1
56	(0,0),(0,1),(2,0),(3,1)	(11,11,12,12)	1
57	(0,0),(0,1),(2,0),(3,4)	(11,12,12,15)	1
58	(0,0),(0,1),(3,0),(4,1)	(12,12,14,14)	1
59	(0,0),(0,1),(0,2),(2,1)	(1,11,11,15)	2
60	(0,0),(0,1),(0,2),(2,0)	(1,11,11,16)	2
61	(0,0),(0,1),(0,5),(2,0)	(4,11,11,14)	2
62	(0,0),(0,1),(0,5),(2,1)	(4,11,11,20)	2
63	(0,0),(0,1),(0,6),(2,1)	(5,5,11,11)	2
64	(0,0),(0,1),(0,6),(2,0)	(5,22,11,11)	2
65	(0,0),(0,1),(0,3),(2,0)	(2,11,15,18)	2
66	(0,0),(0,1),(0,3),(2,1)	(2,11,16,18)	2
67	(0,0),(0,1),(0,4),(2,1)	(3,11,14,18)	2
68	(0,0),(0,1),(0,4),(2,0)	(3,11,20,18)	2
69	(0,0),(0,1),(2,0),(4,6)	(23,11,11,15)	2
70	(0,0),(0,1),(2,0),(4,0)	(6,11,11,14)	2
71	(0,0),(0,1),(2,0),(6,6)	(22,11,15,17)	2
72	(0,0),(0,1),(2,0),(6,0)	(7,11,14,17)	2
73*	(0,0),(0,1),(2,0),(2,1)	(11,11,11,11)	2

Four-Element Motif Classes—Continued			
Class Nr.	*Representative*	*Class Weight*	*Volume*
74*	(0,0),(0,1),(2,0),(2,5)	(11,11,11,11)	2
75*	(0,0),(0,1),(2,0),(2,7)	(11,11,11,11)	2
76*	(0,0),(0,1),(2,0),(2,11)	(11,11,11,11)	2
77*	(0,0),(0,1),(2,0),(6,5)	(11,11,11,19)	2
78*	(0,0),(0,1),(2,0),(6,11)	(11,11,11,19)	2
79*	(0,0),(0,1),(2,0),(4,7)	(11,11,15,15)	2
80*	(0,0),(0,1),(2,0),(8,11)	(11,11,15,15)	2
81*	(0,0),(0,1),(2,0),(2,2)	(11,11,15,16)	2
82*	(0,0),(0,1),(2,0),(8,10)	(11,11,15,16)	2
83*	(0,0),(0,1),(2,0),(4,1)	(11,11,14,14)	2
84*	(0,0),(0,1),(2,0),(8,5)	(11,11,14,14)	2
85*	(0,0),(0,1),(2,0),(8,4)	(11,11,14,20)	2
86*	(0,0),(0,1),(2,0),(2,4)	(11,11,14,20)	2
87*	(0,0),(0,1),(2,0),(6,7)	(11,11,17,17)	2
88*	(0,0),(0,1),(2,0),(6,1)	(11,11,17,17)	2
89*	(0,0),(0,1),(2,0),(2,9)	(11,11,18,18)	2
90*	(0,0),(0,1),(2,0),(2,3)	(11,11,18,18)	2
91	(0,0),(0,1),(2,0),(4,3)	(11,15,15,19)	2
92	(0,0),(0,1),(2,0),(4,2)	(11,15,16,19)	2
93	(0,0),(0,1),(2,0),(4,9)	(11,14,14,19)	2
94	(0,0),(0,1),(2,0),(4,8)	(11,14,20,19)	2
95	(0,0),(0,1),(4,2),(6,1)	(15,15,17,17)	2
96	(0,0),(0,1),(4,2),(6,4)	(15,16,18,18)	2
97	(0,0),(0,1),(4,0),(6,1)	(14,14,17,17)	2
98	(0,0),(0,1),(4,0),(6,4)	(14,20,18,18)	2
99	(0,0),(0,1),(0,2),(3,0)	(1,13,12,18)	3
100	(0,0),(0,1),(0,2),(3,1)	(1,12,12,19)	3
101	(0,0),(0,1),(0,5),(3,0)	(4,4,12,12)	3
102	(0,0),(0,1),(0,4),(3,2)	(4,3,13,12)	3
103	(0,0),(0,1),(0,6),(3,2)	(5,13,13,17)	3
104	(0,0),(0,1),(0,6),(3,1)	(5,12,12,17)	3
105	(0,0),(0,1),(0,6),(3,0)	(5,12,12,25)	3
106	(0,0),(0,1),(0,3),(3,2)	(2,13,12,19)	3
107	(0,0),(0,1),(0,3),(3,1)	(2,12,12,18)	3
108	(0,0),(0,1),(0,3),(3,0)	(2,12,21,18)	3
109	(0,0),(0,1),(0,4),(3,0)	(3,3,12,12)	3
110	(0,0),(0,1),(0,4),(3,1)	(3,3,12,21)	3
111	(0,0),(0,1),(3,0),(6,0)	(8,12,12,17)	3
112	(0,0),(0,3),(0,6),(3,0)	(8,21,21,25)	3

Four-Element Motif Classes—Continued			
Class Nr.	*Representative*	*Class Weight*	*Volume*
113*	(0,0),(0,1),(3,0),(3,5)	(13,13,12,12)	3
114*	(0,0),(0,1),(3,0),(3,11)	(13,13,12,12)	3
115*	(0,0),(0,1),(3,0),(9,11)	(13,13,12,12)	3
116	(0,0),(0,1),(3,2),(3,8)	(13,13,17,17)	3
117	(0,0),(0,1),(3,0),(3,2)	(13,12,19,18)	3
118*	(0,0),(0,1),(3,0),(9,7)	(12,12,12,12)	3
119*	(0,0),(0,1),(3,0),(3,7)	(12,12,12,12)	3
120*	(0,0),(0,1),(3,0),(3,1)	(12,12,12,12)	3
121*	(0,0),(0,1),(3,0),(9,3)	(12,12,12,21)	3
122*	(0,0),(0,1),(3,0),(3,3)	(12,12,12,21)	3
123	(0,0),(0,1),(3,0),(6,5)	(12,12,19,19)	3
124	(0,0),(0,1),(3,0),(6,1)	(12,12,17,17)	3
125	(0,0),(0,1),(3,0),(3,6)	(12,12,17,25)	3
126	(0,0),(0,1),(3,0),(3,10)	(12,12,18,18)	3
127	(0,0),(0,1),(3,0),(6,9)	(12,21,18,18)	3
128*	(0,0),(0,3),(3,0),(3,3)	(21,21,21,21)	3
129*	(0,0),(0,3),(3,0),(9,9)	(21,21,21,21)	3
130	(0,0),(0,3),(3,0),(3,6)	(21,21,25,25)	3
131	(0,0),(0,1),(0,2),(4,3)	(1,15,15,15)	4
132	(0,0),(0,1),(0,2),(4,1)	(1,15,14,14)	4
133	(0,0),(0,1),(0,2),(4,0)	(1,15,14,20)	4
134	(0,0),(0,1),(0,5),(4,2)	(4,15,15,14)	4
135	(0,0),(0,1),(0,5),(4,3)	(4,15,15,20)	4
136	(0,0),(0,1),(0,5),(4,0)	(4,14,14,14)	4
137	(0,0),(0,1),(0,5),(4,1)	(4,14,14,24)	4
138	(0,0),(0,1),(0,6),(4,3)	(5,5,15,15)	4
139	(0,0),(0,1),(0,6),(4,1)	(5,5,14,14)	4
140	(0,0),(0,1),(0,6),(4,0)	(5,7,15,14)	4
141	(0,0),(0,1),(0,3),(4,2)	(2,2,15,15)	4
142	(0,0),(0,1),(0,3),(4,1)	(2,2,14,20)	4
143	(0,0),(0,1),(0,3),(4,0)	(2,3,15,14)	4
144	(0,0),(0,1),(0,3),(4,3)	(2,3,15,20)	4
145	(0,0),(0,1),(0,4),(4,1)	(3,3,14,14)	4
146	(0,0),(0,1),(0,4),(4,0)	(3,3,14,24)	4
147	(0,0),(0,2),(2,0),(6,10)	(23,16,16,16)	4
148	(0,0),(0,2),(2,0),(4,4)	(23,16,20,20)	4
149	(0,0),(0,1),(4,0),(8,6)	(6,15,15,14)	4
150	(0,0),(0,2),(0,4),(2,0)	(6,16,16,20)	4
151	(0,0),(0,2),(0,4),(4,2)	(6,20,20,20)	4

Four-Element Motif Classes—Continued			
Class Nr.	*Representative*	*Class Weight*	*Volume*
152	(0,0),(0,2),(0,4),(4,0)	(6,20,20,24)	4
153	(0,0),(0,2),(2,0),(2,6)	(22,22,16,16)	4
154	(0,0),(0,2),(4,0),(6,2)	(22,22,20,20)	4
155	(0,0),(0,2),(0,6),(2,0)	(22,7,16,20)	4
156	(0,0),(0,2),(0,6),(4,0)	(7,7,20,20)	4
157	(0,0),(0,2),(0,6),(4,2)	(7,7,20,24)	4
158	(0,0),(0,1),(4,0),(8,0)	(9,14,14,14)	4
159	(0,0),(0,2),(4,0),(8,0)	(9,20,20,20)	4
160	(0,0),(0,4),(0,8),(4,0)	(9,24,24,24)	4
161*	(0,0),(0,1),(4,2),(4,7)	(15,15,15,15)	4
162*	(0,0),(0,1),(4,2),(4,3)	(15,15,15,15)	4
163*	(0,0),(0,1),(4,0),(4,7)	(15,15,14,14)	4
164*	(0,0),(0,1),(4,0),(4,11)	(15,15,14,14)	4
165*	(0,0),(0,1),(4,0),(4,2)	(15,15,14,20)	4
166*	(0,0),(0,1),(4,0),(4,10)	(15,15,14,20)	4
167*	(0,0),(0,1),(4,0),(4,1)	(14,14,14,14)	4
168*	(0,0),(0,1),(4,0),(4,5)	(14,14,14,14)	4
169	(0,0),(0,1),(4,0),(4,4)	(14,14,14,24)	4
170	(0,0),(0,2),(2,0),(2,2)	(16,16,16,16)	4
171*	(0,0),(0,2),(2,0),(8,8)	(16,16,20,20)	4
172*	(0,0),(0,2),(2,0),(2,4)	(16,16,20,20)	4
173*	(0,0),(0,2),(4,0),(4,2)	(20,20,20,20)	4
174*	(0,0),(0,2),(4,0),(4,10)	(20,20,20,20)	4
175	(0,0),(0,2),(4,0),(4,4)	(20,20,20,24)	4
176	(0,0),(0,4),(4,0),(4,4)	(24,24,24,24)	4
177	(0,0),(0,1),(0,2),(6,1)	(1,1,17,17)	6
178	(0,0),(0,1),(0,2),(6,3)	(1,2,19,18)	6
179	(0,0),(0,1),(0,2),(6,4)	(1,23,18,18)	6
180	(0,0),(0,1),(0,2),(6,0)	(1,22,19,17)	6
181	(0,0),(0,1),(0,5),(6,0)	(4,4,17,17)	6
182	(0,0),(0,1),(0,4),(6,5)	(4,3,19,18)	6
183	(0,0),(0,1),(0,5),(6,3)	(4,6,18,18)	6
184	(0,0),(0,1),(0,5),(6,1)	(4,7,19,17)	6
185	(0,0),(0,1),(0,6),(6,5)	(5,5,19,19)	6
186	(0,0),(0,1),(0,6),(6,1)	(5,5,17,17)	6
187	(0,0),(0,1),(0,6),(6,2)	(5,22,19,19)	6
188	(0,0),(0,1),(0,6),(6,4)	(5,22,18,18)	6
189	(0,0),(0,1),(0,6),(6,3)	(5,8,18,18)	6
190	(0,0),(0,1),(0,6),(6,0)	(5,26,17,17)	6

Four-Element Motif Classes—Continued			
Class Nr.	*Representative*	*Class Weight*	*Volume*
191	(0,0),(0,1),(0,3),(6,0)	(2,2,17,25)	6
192	(0,0),(0,1),(0,3),(6,4)	(2,2,18,18)	6
193	(0,0),(0,1),(0,3),(6,5)	(2,23,19,18)	6
194	(0,0),(0,1),(0,3),(6,1)	(2,22,17,18)	6
195	(0,0),(0,1),(0,3),(6,3)	(2,22,18,25)	6
196	(0,0),(0,1),(0,4),(6,1)	(3,3,17,25)	6
197	(0,0),(0,1),(0,4),(6,3)	(3,3,18,18)	6
198	(0,0),(0,1),(0,4),(6,2)	(3,6,19,18)	6
199	(0,0),(0,1),(0,4),(6,0)	(3,7,17,18)	6
200	(0,0),(0,1),(0,4),(6,4)	(3,7,18,25)	6
201	(0,0),(0,1),(6,3),(6,9)	(22,8,18,18)	6
202	(0,0),(0,3),(0,6),(6,3)	(8,8,25,25)	6
203	(0,0),(0,3),(0,6),(6,0)	(8,26,25,25)	6
204*	(0,0),(0,1),(6,0),(6,5)	(19,19,17,17)	6
205*	(0,0),(0,1),(6,0),(6,11)	(19,19,17,17)	6
206*	(0,0),(0,1),(6,2),(6,3)	(19,19,18,18)	6
207*	(0,0),(0,1),(6,2),(6,5)	(19,19,18,18)	6
208*	(0,0),(0,1),(6,0),(6,1)	(17,17,17,17)	6
209*	(0,0),(0,1),(6,0),(6,7)	(17,17,17,17)	6
210*	(0,0),(0,1),(6,0),(6,3)	(17,18,18,25)	6
211*	(0,0),(0,1),(6,0),(6,9)	(17,18,18,25)	6
212*	(0,0),(0,1),(6,3),(6,4)	(18,18,18,18)	6
213*	(0,0),(0,1),(6,3),(6,10)	(18,18,18,18)	6
214*	(0,0),(0,3),(6,0),(6,3)	(25,25,25,25)	6
215*	(0,0),(0,3),(6,0),(6,9)	(25,25,25,25)	6

O.5 Three Tone Motifs in $OnPiMod_{5,12}$

Refer to the discussion in Section 11.3.8 for the entries of this table. The order of these representatives is a historical one.

Three-Element Motif Classes in $OnPiMod_{5,12}$		
Cl. Nr.	*Representative in* $\mathbb{Z}_5 \times \mathbb{Z}_{12}$	*Kernel*
1	(0,0),(1,0),(0,4)	$\mathbb{Z}_4^2 \times (\mathbb{Z}_3 \times 0) \times (0 \times \mathbb{Z}_5)$
2	(0,0),(1,0),(4,4)	$\mathbb{Z}_4^2 \times (\mathbb{Z}_3 \times 0) \times \mathbb{Z}_5.(1,1)$
3	(0,0),(1,0),(3,4)	$\mathbb{Z}_4^2 \times (\mathbb{Z}_3 \times 0) \times \mathbb{Z}_5.(2,1)$
4	(0,0),(0,4),(1,8)	$\mathbb{Z}_4^2 \times \mathbb{Z}_3.(1,1) \times (\mathbb{Z}_5 \times 0)$
5	(0,0),(1,4),(4,8)	$\mathbb{Z}_4^2 \times \mathbb{Z}_3.(1,1) \times \mathbb{Z}_5.(1,1)$
6	(0,0),(1,0),(0,10)	$(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (\mathbb{Z}_3 \times 0) \times (0 \times \mathbb{Z}_5)$
7	(0,0),(1,0),(4,10)	$(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (\mathbb{Z}_3 \times 0) \times \mathbb{Z}_5.(1,1)$
8	(0,0),(1,0),(3,10)	$(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (\mathbb{Z}_3 \times 0) \times \mathbb{Z}_5.(2,1)$
9	(0,0),(0,4),(1,2)	$(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times \mathbb{Z}_3.(1,1) \times (\mathbb{Z}_5 \times 0)$
10	(0,0),(1,4),(0,2)	$(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times \mathbb{Z}_3.(1,1) \times (0 \times \mathbb{Z}_5)$
11	(0,0),(1,4),(4,2)	$(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times \mathbb{Z}_3.(1,1) \times \mathbb{Z}_5.(1,1)$
12	(0,0),(1,4),(3,2)	$(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times \mathbb{Z}_3.(1,1) \times \mathbb{Z}_5.(2,1)$
13	(0,0),(0,4),(1,6)	$(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (0 \times \mathbb{Z}_3) \times (\mathbb{Z}_5 \times 0)$
14	(0,0),(1,4),(0,6)	$(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (0 \times \mathbb{Z}_3) \times (0 \times \mathbb{Z}_5)$
15	(0,0),(1,4),(4,6)	$(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(1,1)$
16	(0,0),(1,4),(3,6)	$(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(2,1)$
17	(0,0),(3,4),(1,6)	$(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(1,2)$
18	(0,0),(1,4),(1,6)	$(\mathbb{Z}_4 \times 2\mathbb{Z}_4) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(4,1)$
19	(0,0),(1,0),(0,1)	$(\mathbb{Z}_4 \times 0) \times (\mathbb{Z}_3 \times 0) \times (0 \times \mathbb{Z}_5)$
20	(0,0),(1,0),(4,1)	$(\mathbb{Z}_4 \times 0) \times (\mathbb{Z}_3 \times 0) \times \mathbb{Z}_5.(1,1)$
21	(0,0),(1,0),(3,1)	$(\mathbb{Z}_4 \times 0) \times (\mathbb{Z}_3 \times 0) \times \mathbb{Z}_5.(2,1)$
22	(0,0),(0,4),(1,5)	$(\mathbb{Z}_4 \times 0) \times \mathbb{Z}_3.(1,1) \times (\mathbb{Z}_5 \times 0)$
23	(0,0),(1,4),(0,5)	$(\mathbb{Z}_4 \times 0) \times \mathbb{Z}_3.(1,1) \times (0 \times \mathbb{Z}_5)$
24	(0,0),(1,4),(4,5)	$(\mathbb{Z}_4 \times 0) \times \mathbb{Z}_3.(1,1) \times \mathbb{Z}_5.(1,1)$
25	(0,0),(1,4),(3,5)	$(\mathbb{Z}_4 \times 0) \times \mathbb{Z}_3.(1,1) \times \mathbb{Z}_5.(2,1)$
26	(0,0),(0,4),(1,9)	$(\mathbb{Z}_4 \times 0) \times (0 \times \mathbb{Z}_3) \times (\mathbb{Z}_5 \times 0)$
27	(0,0),(1,4),(0,9)	$(\mathbb{Z}_4 \times 0) \times (0 \times \mathbb{Z}_3) \times (0 \times \mathbb{Z}_5)$
28	(0,0),(1,4),(4,9)	$(\mathbb{Z}_4 \times 0) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(1,1)$
29	(0,0),(1,4),(3,9)	$(\mathbb{Z}_4 \times 0) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(2,1)$
30	(0,0),(3,4),(1,9)	$(\mathbb{Z}_4 \times 0) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(1,2)$
31	(0,0),(1,4),(1,9)	$(\mathbb{Z}_4 \times 0) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(4,1)$
32	(0,0),(0,6),(1,1)	$\mathbb{Z}_4.(1,2) \times (\mathbb{Z}_3 \times 0) \times (\mathbb{Z}_5 \times 0)$

Three-Element Motif Classes in $OnPiMod_{5,12}$—continued		
Cl. Nr.	*Representative in* $\mathbb{Z}_5 \times \mathbb{Z}_{12}$	*Kernel*
33	(0,0),(1,6),(0,1)	$\mathbb{Z}_4.(1,2) \times (\mathbb{Z}_3 \times 0) \times (0 \times \mathbb{Z}_5)$
34	(0,0),(1,6),(4,1)	$\mathbb{Z}_4.(1,2) \times (\mathbb{Z}_3 \times 0) \times \mathbb{Z}_5.(1,1)$
35	(0,0),(1,6),(3,1)	$\mathbb{Z}_4.(1,2) \times (\mathbb{Z}_3 \times 0) \times \mathbb{Z}_5.(2,1)$
36	(0,0),(0,10),(1,5)	$\mathbb{Z}_4.(1,2) \times \mathbb{Z}_3.(1,1) \times (\mathbb{Z}_5 \times 0)$
37	(0,0),(1,10),(0,5)	$\mathbb{Z}_4.(1,2) \times \mathbb{Z}_3.(1,1) \times (0 \times \mathbb{Z}_5)$
38	(0,0),(1,10),(4,5)	$\mathbb{Z}_4.(1,2) \times \mathbb{Z}_3.(1,1) \times \mathbb{Z}_5.(1,1)$
39	(0,0),(1,10),(3,5)	$\mathbb{Z}_4.(1,2) \times \mathbb{Z}_3.(1,1) \times \mathbb{Z}_5.(2,1)$
40	(0,0),(0,10),(1,9)	$\mathbb{Z}_4.(1,2) \times (0 \times \mathbb{Z}_3) \times (\mathbb{Z}_5 \times 0)$
41	(0,0),(1,10),(0,9)	$\mathbb{Z}_4.(1,2) \times (0 \times \mathbb{Z}_3) \times (0 \times \mathbb{Z}_5)$
42	(0,0),(4,10),(1,9)	$\mathbb{Z}_4.(1,2) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(1,1)$
43	(0,0),(3,10),(1,9)	$\mathbb{Z}_4.(1,2) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(1,2)$
44	(0,0),(1,10),(3,9)	$\mathbb{Z}_4.(1,2) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(2,1)$
45	(0,0),(1,10),(1,9)	$\mathbb{Z}_4.(1,2) \times (0 \times \mathbb{Z}_3) \times \mathbb{Z}_5.(4,1)$

P

Well-Tempered and Just Modulation Steps

P.1 12-Tempered Modulation Steps

P.1.1 Scale Orbits and Number of Quantized Modulations

In the following table, the exclamation sign (!) in column 6 means that quantization is not possible for every translation quantity p in the notation of Theorem 30.

Orbits and Number of Quantized Modulations			
Class	*# Min. Cadence Sets*	*# Quanta*	*# Quant. Mod.*
38	9	42	54 (!)
38.1	5	20	26
47	6	28	30
47.1	15	66	114
50	7	34	42
50.1	6	36	46
52	5	24	24 (!)
55	6	30	32 (!)
61	10	38	62
62	5	24	24 (!)
39	9	29	93
39.1	6	23	55
40	10	24	108
40.1	7	26	72
41	7	25	75
41.1	6	21	53
42	6	22	54
42.1	7	28	74
43	6	22	57
43.1	7	26	72
44	9	23	89
45	7	21	63

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Orbits and Number of Quant. Mod.—Continued			
Class	*# Min. Cadence Sets*	*# Quanta*	*# Quant. Mod.*
45.1	10	21	105
46	6	26	56
48	10	23	109
48.1	7	28	68
49	7	21	71
49.1	7	26	74
51	9	13	86
53	7	27	67
53.1	9	25	91
54	7	32	71
54.1	21	32	226
56	7	24	70
57	8	21	71
58	18	17	185
59	11	22	101
60	6	21	60

P.1.2 Quanta and Pivots for the Modulations Between Diatonic Major Scales (No.38.1)

Quanta and Pivots for the Modulations Between Diatonic Major Scales				
Transl. p	*Cadence*	*Quantum*	*Modulator*	*Pivots*
1	$\{II, V\}$	●○●●○●●●●●●●	$e^5 11$	$\{II, III, V, VII\}$
1	$\{II, III\}$	●○●●○●●●●●●●	$e^5 11$	$\{II, III, V, VII\}$
2	$\{VII\}$	○●●○●●○●○○○●	$e^6 11$	$\{II, IV, VII\}$
2	$\{II, V\}$	○●●○●●○●○●○●	$e^6 11$	$\{II, IV, V, VII\}$
2	$\{IV, V\}$	○●●○●●○●○●○●	$e^6 11$	$\{II, IV, V, VII\}$
3	$\{II, V\}$	●○●○○●○●●●●●	$e^7 11$	$\{II, III, V, VII\}$
3	$\{II, III\}$	●○●○○●○●●●●●	$e^7 11$	$\{II, III, V, VII\}$
4	$\{VII\}$	○○●●○●●○○●○●	$e^8 11$	$\{II, IV, V, VII\}$
4	$\{IV, V\}$	○●●●●●●●○●○●	$e^8 11$	$\{II, III, V, VII\}$
4	$\{II, III\}$	●●●●○●●●●●○●	$e^8 11$	$\{V, VII\}$
5	$\{VII\}$	○○●○●●○●○○●●	$e^9 11$	$\{II, IV, VII\}$
6	$\{II, III\}$	○●●●●●○●●●●●	e^6	$\{II, III, V, VII\}$
6	$\{IV, V\}$	○●●●●●●●●●○●	$e^{10} 11$	$\{II, IV, V, VII\}$
6	$\{IV, V\}$	●●●●○●●●●●○●	e^6	$\{II, IV, V, VII\}$
6	$\{II, III\}$	●●●●○●○●●●●●	$e^{10} 11$	$\{II, III, V, VII\}$
7	$\{VII\}$	●○●○○●●○○●○●	$e^{11} 11$	$\{III, V, VII\}$
8	$\{VII\}$	○●●○○●○●○○●●	$e^0 11$	$\{II, VII\}$
8	$\{IV, V\}$	○●●●●●○●●●●●	$e^0 11$	$\{II, IV, V, VII\}$
8	$\{II, III\}$	●●●●○●○●○●●●	$e^0 11$	$\{II, III, V, VII\}$
9	$\{II, V\}$	○○●○●●●●●●○●	$e^1 11$	$\{II, IV, V, VII\}$
9	$\{IV, V\}$	○○●○●●●●●●○●	$e^1 11$	$\{II, IV, V, VII\}$
10	$\{VII\}$	●○●●○●○○○●○●	$e^2 11$	$\{III, V, VII\}$
10	$\{II, V\}$	●○●●○●○●○●○●	$e^2 11$	$\{II, III, V, VII\}$
10	$\{II, III\}$	●○●●○●○●○●○●	$e^2 11$	$\{II, III, V, VII\}$
11	$\{II, V\}$	○●●○●●●●●●●●	$e^3 11$	$\{II, IV, V, VII\}$
11	$\{IV, V\}$	○●●○●●●●●●●●	$e^3 11$	$\{II, IV, V, VII\}$

P.1.3 Quanta and Pivots for the Modulations Between Melodic Minor Scales (No.47.1)

The symbol p stands for "translation p"; M stands for "Modulator".

Quanta and Pivots for the Modulations Between Melodic Minor Scales				
p	*Cadence*	*Quantum*	M	*Pivots*
1	$\{II, IV\}, \{IV, VII\}$	●●●●○●●○○●●○	$e^3 11$	$\{II, IV, VII\}$
1	$\{III, VI\}, \{V, VI\}$	●●●●●●○●●○●●	$e^3 11$	$\{I, III, V, VI\}$
1	$\{III, VII\}$	●○○●●○●●●●○●	$e^3 11$	$\{III, V, VII\}$
1	$\{IV, V\}$	●●●●○●●●●●●○	$e^3 11$	$\{II, IV, V, VII\}$
1	$\{II, III\}$	●○○●●●●●●●●●	$e^3 11$	$\{II, III, V, VII\}$
1	$\{I, VII\}$	●●●●●○●●●●○●	$e^3 11$	$\{I, III, V, VII\}$
2	$\{III, V\}, \{III, VII\}, \{II, III\}$	●●○●●●○●○●○●	$e^4 11$	$\{II, III, V, VII\}$
2	$\{II, IV\}, \{II, VI\}, \{I, II\}$	●○●○●●○●○●○●	$e^4 11$	$\{I, II, IV, VI\}$
2	$\{I, III\}, \{III, VI\}, \{III, IV\}$	○●●●○●○●○●○●	$e^4 11$	$\{I, III, IV, VI\}$
3	$\{III, V\}, \{III, VI\}, \{V, VI\}, \{I, V\}$	●○●●○●●●○○●●	$e^5 11$	$\{I, III, V, VI\}$
3	$\{III, V\}, \{III, VI\}, \{V, VI\}, \{I, V\}$	●○●●○●●●○○●●	$e^5 11$	$\{I, III, V, VI\}$
3	$\{III, V\}, \{III, VI\}, \{V, VI\}, \{I, V\}$	●○●●○●●●○○●●	$e^5 11$	$\{I, III, V, VI\}$
3	$\{III, V\}, \{III, VI\}, \{V, VI\}, \{I, V\}$	●○●●○●●●○○●●	$e^5 11$	$\{I, III, V, VI\}$
4	$\{III, V\}$	○●●○●●○○●●●○	$e^6 11$	$\{III, V\}$
4	$\{I, III\}$	○○●●●○○●○○○●	$e^6 11$	$\{I, III\}$
4	$\{II, VI\}, \{I, II\}$	●●●○●●●●○●○●	$e^6 11$	$\{I, II, IV, VI\}$
4	$\{IV, VII\}$	●●●●●●●○○●○○	$e^6 11$	$\{II, IV, VII\}$
4	$\{III, VI\}$	○●●●●●○●○○○●	$e^6 11$	$\{I, III, VI\}$
4	$\{III, VII\}$	●○○●○○●●○●○●	$e^6 11$	$\{III, V, VII\}$
4	$\{V, VI\}$	●●●●●●●●○○○●	$e^6 11$	$\{I, III, V, VI\}$
4	$\{III, IV\}$	○●●●●●○●○●○●	$e^6 11$	$\{I, III, IV, VI\}$
4	$\{II, III\}$	●●○●○●●●○●○●	$e^6 11$	$\{II, III, V, VII\}$
4	$\{I, VII\}$	●○●●●○●●○●○●	$e^6 11$	$\{I, III, V, VII\}$
4	$\{I, III\}$	●○○●●○○●●○○●	e^4	$\{I, III\}$
4	$\{III, V\}$	○○●●○○●●○○●●	e^4	$\{III, V\}$
4	$\{III, VII\}, \{II, III\}$	○●●●○●●●○●●●	e^4	$\{II, III, V, VII\}$
4	$\{III, VI\}, \{III, IV\}$	●●○●●●○●●●○●	e^4	$\{I, III, IV, VI\}$
5	$\{I, II\}, \{I, V\}, \{III, VI\}, \{V, VI\}$	●○●●●●○●●○○●	$e^7 11$	$\{I, III, V, VI\}$
5	$\{II, VI\}, \{I, II\}$	●○●○○●○●●●●●	$e^7 11$	$\{I, II, IV, VI\}$
5	$\{IV, VII\}, \{IV, V\}$	●○●●●●○●○●●○	$e^7 11$	$\{II, IV, V, VII\}$
5	$\{III, VII\}$	●○○●●○○●●●●●	$e^7 11$	$\{III, V, VII\}$
6	$\{III, V\}, \{III, VII\}, \{II, III\}$	●●○●○●○●●●○●	$e^8 11$	$\{II, III, V, VII\}$
6	$\{I, III\}, \{III, VI\}, \{III, IV\}$	○●●●○●●●○●○●	$e^8 11$	$\{I, III, IV, VI\}$
6	$\{I, III\}, \{III, VI\}, \{III, IV\}$	●●○●○●●●○●○●	e^6	$\{I, III, IV, VI\}$

Quanta and Pivots for Melodic Minor Scales—Continued				
p	*Cadence*	*Quantum*	M	*Pivots*
6	$\{III, V\}, \{III, VII\}, \{II, III\}$	○ ● ● ● ○ ● ○ ● ● ● ○ ●	e^6	$\{II, III, V, VII\}$
7	$\{III, V\}, \{I, V\}, \{III, VII\}, \{I, VII\}$	● ○ ● ● ○ ○ ● ● ○ ● ● ●	$e^9 11$	$\{I, III, V, VII\}$
7	$\{II, VI\}, \{I, II\}$	● ○ ● ○ ● ● ○ ● ○ ● ● ●	$e^9 11$	$\{I, II, IV, VI\}$
7	$\{IV, VII\}, \{IV, V\}$	● ○ ● ● ● ● ● ● ○ ● ○ ○	$e^9 11$	$\{II, IV, V, VII\}$
7	$\{III, VI\}$	○ ○ ● ● ● ● ● ● ○ ○ ● ●	$e^9 11$	$\{I, III, VI\}$
8	$\{III, V\}$	● ○ ○ ● ○ ○ ○ ● ○ ○ ● ●	$e^{10} 11$	$\{III, V\}$
8	$\{I, III\}$	○ ○ ● ● ○ ○ ○ ● ● ○ ○ ●	$e^{10} 11$	$\{I, III\}$
8	$\{II, VI\}$	● ● ● ○ ○ ● ○ ○ ● ● ● ●	$e^{10} 11$	$\{II, IV, VI\}$
8	$\{IV, VII\}, \{IV, V\}$	● ● ● ● ○ ● ○ ● ● ● ● ○	$e^{10} 11$	$\{II, IV, V, VII\}$
8	$\{III, VI\}$	○ ○ ● ● ○ ● ○ ● ● ○ ○ ●	$e^{10} 11$	$\{I, III, VI\}$
8	$\{III, VII\}$	● ● ○ ● ○ ○ ○ ● ○ ● ● ●	$e^{10} 11$	$\{III, V, VII\}$
8	$\{V, VI\}$	● ○ ● ● ○ ● ○ ● ● ○ ● ●	$e^{10} 11$	$\{I, III, V, VI\}$
8	$\{III, IV\}$	○ ● ● ● ○ ● ○ ● ● ● ○ ●	$e^{10} 11$	$\{I, III, IV, VI\}$
8	$\{II, III\}$	● ● ○ ● ○ ● ○ ● ○ ● ● ●	$e^{10} 11$	$\{II, III, V, VII\}$
8	$\{I, VII\}$	● ● ● ● ○ ○ ○ ● ● ● ● ●	$e^{10} 11$	$\{I, III, V, VII\}$
8	$\{I, III\}$	● ○ ○ ● ● ○ ○ ● ● ○ ○ ●	e^8	$\{I, III\}$
8	$\{III, V\}$	○ ○ ● ● ○ ○ ● ● ○ ○ ● ●	e^8	$\{III, V\}$
8	$\{III, VII\}, \{II, III\}$	○ ● ● ● ○ ● ● ● ○ ● ● ●	e^8	$\{II, III, V, VII\}$
8	$\{III, VI\}, \{II, IV\}$	● ● ○ ● ● ● ○ ● ● ● ○ ●	e^8	$\{I, III, IV, VI\}$
9	$\{III, V\}$	● ○ ○ ● ● ○ ○ ● ● ○ ○ ●	$e^{11} 11$	$\{III, V\}$
9	$\{II, IV\}, \{II, VI\}$	● ○ ● ○ ○ ● ● ○ ○ ● ○ ●	$e^{11} 11$	$\{II, IV, VI\}$
9	$\{I, III\}, \{I, V\}, \{III, VII\}, \{I, VII\}$	● ○ ● ● ● ○ ○ ● ● ● ○ ●	$e^{11} 11$	$\{I, III, V, VII\}$
9	$\{I, II\}$	● ○ ● ○ ● ● ● ● ○ ● ○ ●	$e^{11} 11$	$\{I, III, IV, VI\}$
10	$\{II, V\}, \{III, VII\}, \{II, III\}$	● ● ○ ● ○ ● ○ ● ○ ● ○ ●	$e^0 11$	$\{II, III, V, VII\}$
10	$\{II, IV\}, \{IV, VII\}, \{IV, V\}$	● ○ ● ● ○ ● ○ ● ○ ● ● ○	$e^0 11$	$\{II, IV, V, VII\}$
10	$\{I, III\}, \{III, VI\}, \{III, IV\}$	○ ● ● ● ○ ● ○ ● ○ ● ● ●	$e^0 11$	$\{I, III, IV, VI\}$
11	$\{II, IV\}, \{II, VI\}$	● ● ● ○ ● ● ○ ○ ● ● ○ ●	$e^1 11$	$\{I, IV, VI\}$
11	$\{III, VI\}$	○ ○ ● ● ○ ● ● ● ● ○ ● ●	$e^1 11$	$\{I, III, VI\}$
11	$\{III, VII\}, \{I, VII\}$	● ● ● ● ○ ● ● ○ ● ● ● ●	$e^1 11$	$\{I, III, V, VII\}$
11	$\{V, VI\}$	● ● ● ● ○ ● ● ● ● ○ ● ●	$e^1 11$	$\{I, III, V, VI\}$
11	$\{III, IV\}$	○ ○ ● ● ● ● ● ● ● ● ● ●	$e^1 11$	$\{I, III, IV, VI\}$
11	$\{I, II\}$	● ● ● ○ ● ● ● ● ● ● ○ ●	$e^1 11$	$\{I, II, IV, VI\}$

P.1.4 Quanta and Pivots for the Modulations Between Harmonic Minor Scales (No.54.1)

For this table, we need a numbering of the 21 minimal cadence sets:

$0 = \{II, VII\}$ $1 = \{I, III\}$ $2 = \{II, IV\}$ $3 = \{III, V\}$ $4 = \{IV, VI\}$
$5 = \{V, VII\}$ $6 = \{I, VI\}$ $7 = \{IV, VII\}$ $8 = \{I, V\}$ $9 = \{II, VI\}$
$10 = \{III, VII\}$ $11 = \{I, IV\}$ $12 = \{II, V\}$ $13 = \{III, VI\}$ $14 = \{VI, VII\}$
$15 = \{I, VII\}$ $16 = \{I, II\}$ $17 = \{II, III\}$ $18 = \{III, IV\}$ $19 = \{IV, V\}$
$20 = \{V, VI\}$

Quanta and Pivots for the Modulations Between Harmonic Minor Scales			
Transl. p	*Cadence Nr.*	*Quantum*	*Pivots*
3/9	1,3,6,8,10,11,15-20	○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○	$\{I, II, III, IV, V, VI, VII\}$
3/9	2,4,7,9,14	● ○ ● ● ○ ● ● ○ ● ● ○ ●	$\{II, IV, VI, VII\}$
3/9	5,12	○ ● ● ○ ● ● ○ ● ● ○ ● ●	$\{II, V, VII\}$
4/8	0,7,12,14-17,19	○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○	$\{I, II, III, IV, V, VI, VII\}$
4/8	1,6,13	● ○ ○ ● ● ○ ○ ● ● ○ ○ ●	$\{I, II, VI\}$
4/8	2	● ● ● ○ ● ● ● ○ ● ● ● ○	$\{II, IV\}$
4/8	3	○ ○ ● ● ○ ○ ● ● ○ ○ ● ●	$\{III, V\}$
4/8	4,11,18	● ● ○ ● ● ● ○ ● ● ● ○ ●	$\{I, III, IV, VI\}$
4/8	5,10	○ ● ● ● ○ ● ● ● ○ ● ● ●	$\{III, V, VII\}$
4/8	8,20	● ○ ● ● ● ○ ● ● ● ○ ● ●	$\{I, III, V, VI\}$
6	2,7	● ○ ● ○ ○ ● ● ○ ● ○ ○ ●	$\{II, IV, VII\}$
6	3,10,17	○ ● ● ● ○ ● ○ ● ● ● ○ ●	$\{II, III, V, VII\}$
6	4,9,14	● ○ ● ● ○ ● ● ○ ● ● ○ ●	$\{II, IV, VI, VII\}$
6	5,12	○ ● ● ○ ○ ● ○ ● ● ○ ○ ●	$\{II, V, VII\}$
6	8,11,13,15,16,18	● ● ● ● ○ ● ● ● ● ● ○ ●	$\{I, II, II, IV, V, VI, VII\}$
6	19	● ● ● ○ ○ ● ● ● ● ○ ○ ●	$\{II, IV, V, VII\}$
1,2,5,7,10,11	any cadence set	● ● ● ● ● ● ● ● ● ● ● ●	$\{I, II, II, IV, V, VI, VII\}$

P.1.5 Examples of 12-Tempered Modulations for All Fourth Relations

Examples of Modulations for all Fourth Relations			
Start → *Target*	*Neutral*	*Pivots*	*Cadence*
$C \rightsquigarrow F$	I_C	$VII_F \cup II_F$	I_F, IV_F, V_F, I_F
$C \rightsquigarrow B_\flat$	I_C, V_C	$III_{B_\flat}, V_{B_\flat} \cup VII_{B_\flat}$	$VII_{B_\flat}, I_{B_\flat}$
$C \rightsquigarrow E_\flat$	I_C, V_C	$II_{E_\flat}, V_{E_\flat} \cup VII_{E_\flat}$	$V_{E_\flat} \cup VII_{E_\flat}, I_{E_\flat}$
$C \rightsquigarrow A_\flat$	I_C, IV_C	$II_{A_\flat} \cup VII_{A_\flat}$	$IV_{A_\flat}, V_{A_\flat}, I_{A_\flat}$
$C \rightsquigarrow D_\flat$	I_C	$III_{D_\flat}, II_{D_\flat} \cup VII_{D_\flat}$	$II_{D_\flat}, V_{D_\flat}, I_{D_\flat}$
$C \rightsquigarrow G_\flat$	$I_C, IV_C \cup VI_C, VI_C \cup VII_C$	$II_{G_\flat} \cup VII_{G_\flat}, II_{G_\flat}$	$V_{G_\flat} \cup VII_{G_\flat}, I_{G_\flat}$

Examples of Modulations for all Fourth Relations—Continued			
Start → *Target*	*Neutral*	*Pivots*	*Cadence*
$C \rightsquigarrow B$	I_C, V_C, II_C	$IV_B, II_B \cup VII_B$	IV_B, V_B, I_B
$C \rightsquigarrow E$	I_C, VI_C	$V_E \cup VII_E$	IV_E, V_E, I_E
$C \rightsquigarrow A$	I_C, V_C	IV_A	$IV_A \cup VII_A, V_A, I_A$
$C \rightsquigarrow D$	I_C, VI_C, V_C	$VII_D \cup V_D$	II_D, V_D, I_D
$C \rightsquigarrow G$	I_C, V_C	III_G	II_G, V_G, I_G

P.2 2-3-5-Just Modulation Steps

The following tables show data for modulations from C-tonic.

P.2.1 Modulation Steps Between Just Major Scales

Here, we have the two modulators

$$\Phi_1 = e^b, \tag{P.1}$$
$$\Phi_2 = e^b.A. \tag{P.2}$$

The numbering of minimal cadence sets is the one used in Formula (26.2). The tonics D^* and $B_\flat^*$ are the usual third comma shifted representatives of D and B-flat.

Pivots for the Modulations Between Just Major Scales				
Translation	*Target Tonic*	*Modulator*	*Cadence*	*Pivots*
$(1,0)$	G	Φ_2	5	$\{V, VII\}$
$(-1,0)$	F	Φ_2	1	$\{II, IV\}$
$(2,0)$	D	Φ_2	1	$\{II, IV\}$
$(-2,0)$	$B_\flat$	Φ_2	5	$\{V, VII\}$
$(0,1)$	E	Φ_2	1	$\{II, V, VII\}$
$(0,1)$	E	Φ_1	5	$\{II, V, VII\}$
$(0,-1)$	$A_\flat$	Φ_2	5	$\{II, IV, VII\}$
$(0,-1)$	$A_\flat$	Φ_1	5	$\{II, V, VII\}$
$(1,1)$	B	Φ_2	1	$\{II, V, VI\}$
$(-1,-1)$	$D_\flat$	Φ_2	5	$\{II, V, VII\}$
$(1,-1)$	$E_\flat$	Φ_2	1	$\{II, V, VII\}$
$(1,-1)$	$E_\flat$	Φ_1	1	$\{II, IV, VII\}$
$(-1,1)$	A	Φ_2	5	$\{II, IV, VII\}$
$(-1,1)$	A	Φ_1	1	$\{II, IV, VII\}$
$(-2,1)$	D^*	Φ_2	5	$\{III, V, VII\}$
$(2,-1)$	$B_\flat^*$	Φ_2	1	$\{II, IV, VI\}$

P.2.2 Modulation Steps Between Natural Minor Scales

We have the two modulators

$$\Phi_1 = e^b,$$
$$\Phi_2 = e^b.A.$$

The minimal cadence sets are these:

$$J_1 = \{VII\}, J_2 = \{III, VI\}, J_3 = \{V, VI\},$$
$$J_4 = \{IV, V\}, J_5 = \{II\}, J_6 = \{III, IV\}.$$

Pivots for the Modulations Between Natural Minor Scales

Translation	*Target Tonic*	*Modulator*	*Cadence*	*Pivots*
$(-1, 0)$	F	Φ_2	5	$\{II, IV\}$
$(1, 0)$	G	Φ_2	1	$\{V, VII\}$
$(-2, 0)$	$B_\flat$	Φ_2	1	$\{V, VII\}$
$(2, 0)$	D	Φ_2	5	$\{II, IV\}$
$(0, -1)$	$A_\flat$	Φ_2	1	$\{II, IV, VII\}$
$(0, -1)$	$A_\flat$	Φ_1	5	$\{II, IV, VII\}$
$(0, 1)$	E	Φ_1	5	$\{II, IV, VII\}$
$(0, 1)$	E	Φ_2	5	$\{II, V, VII\}$
$(-1, -1)$	$D_\flat$	Φ_2	1	$\{III, V, VII\}$
$(1, 1)$	B	Φ_2	5	$\{II, IV, VI\}$
$(1, -1)$	$E_\flat$	Φ_1	1	$\{II, V, VII\}$
$(1, -1)$	$E_\flat$	Φ_2	5	$\{II, V, VII\}$
$(-1, 1)$	A	Φ_1	1	$\{II, V, VII\}$
$(-1, 1)$	A	Φ_2	1	$\{II, IV, VII\}$
$(2, -1)$	$B_\flat^*$	Φ_2	5	$\{II, IV, VI\}$
$(-2, 1)$	D^*	Φ_2	1	$\{III, V, VII\}$

P.2.3 Modulation Steps from Natural Minor to Major Scales

We have the two modulators

$$\Phi_1 = e^b.A,$$
$$\Phi_2 = e^b.B, B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

The minimal cadence sets are the same as for major scales.

Pivots for the Modulations From Natural Minor to Major Scales				
Translation	*Target Tonic*	*Modulator*	*Cadence*	*Pivots*
$(-2,0)$	$B_\flat$	Φ_1	5	$\{V, VII\}$
$(-1,0)$	F	Φ_1	1	$\{II, IV\}$
$(1,0)$	G	Φ_1	5	$\{V, VII\}$
$(2,0)$	D	Φ_1	1	$\{II, IV\}$
$(-1,-1)$	$D_\flat$	Φ_1	5	$\{II, V, VII\}$
$(0,-1)$	$A_\flat$	Φ_1	5	$\{II, IV, VII\}$
$(1,-1)$	$E_\flat$	Φ_1	1	$\{II, V, VII\}$
$(2,-1)$	$B_\flat^*$	Φ_1	1	$\{II, IV, VI\}$

P.2.4 Modulation Steps from Major to Natural Minor Scales

We have the two modulators (same as above).

$$\Phi_1 = e^b.A,$$
$$\Phi_2 = e^b.B, B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

The minimal cadence sets are those of minor scales.

Pivots for the Modulations From Major to Natural Minor Scales				
Translation	*Target Tonic*	*Modulator*	*Cadence*	*Pivots*
$(-2,0)$	$B_\flat$	Φ_1	1	$\{V, VII\}$
$(-1,0)$	F	Φ_1	5	$\{II, IV\}$
$(1,0)$	G	Φ_1	1	$\{V, VII\}$
$(2,0)$	D	Φ_1	5	$\{II, IV\}$
$(-2,1)$	D	Φ_1	1	$\{III, V, VII\}$
$(-1,1)$	A	Φ_1	1	$\{II, IV, VII\}$
$(0,1)$	E	Φ_1	5	$\{II, V, VII\}$
$(1,1)$	B	Φ_1	5	$\{II, IV, VI\}$

P.2.5 Modulation Steps Between Harmonic Minor Scales

We have the unique translation modulator

$$\Phi = e^b.$$

The minimal cadence sets are these:

$$J_1 = \{III\}, J_2 = \{II\}, J_3 = \{VII\}, J_4 = \{I, IV\}, J_5 = \{I, V\},$$
$$J_6 = \{I, VI\}, J_7 = \{IV, V\}, J_8 = \{IV, VI\}, J_9 = \{V, VI\}.$$

Pivots for the Modulations Between Harmonic Minor Scales				
Translation	*Target Tonic*	*Modulator*	*Cadence*	*Pivots*
$(3,0)$	A^*	Φ	8	$\{II, IV, VI\}$
$(-3,0)$	$E_\flat^*$	Φ	8	$\{II, IV, VI\}$
$(0,1)$	E	Φ	2	$\{II, IV\}$
$(0,1)$	E	Φ	4	$\{I, III, IV, VI\}$
$(0,1)$	E	Φ	8	$\{I, III, IV, VI\}$
$(0,-1)$	$A_\flat$	Φ	2	$\{II, IV\}$
$(0,-1)$	$A_\flat$	Φ	4	$\{I, III, IV, VI\}$
$(0,-1)$	$A_\flat$	Φ	8	$\{I, III, IV, VI\}$
$(2,1)$	$f_\sharp$	Φ	8	$\{II, IV, VI, VII\}$
$(-2,-1)$	$G_\flat$	Φ	8	$\{II, IV, VI, VII\}$
$(0,2)$	$G_\sharp$	Φ	4	$\{I, III, IV, VI\}$
$(0,-2)$	$F_\flat$	Φ	4	$\{I, III, IV, VI\}$
$(1,2)$	$D_\sharp$	Φ	4	$\{I, III, IV, VI\}$
$(-1,-2)$	$B_{\flat,\flat}$	Φ	4	$\{I, III, IV, VI\}$
$(1,0)$	G	Φ	1,5,7,9	$\{I \ldots VII\}$
$(-1,0)$	F	Φ	1,5,7,9	$\{I \ldots VII\}$
$(2,0)$	D	Φ	9	$\{I \ldots VII\}$
$(-2,0)$	$B_\flat$	Φ	9	$\{I \ldots VII\}$
$(3,0)$	A^*	Φ	9	$\{I \ldots VII\}$
$(-3,0)$	$E_\flat^*$	Φ	9	$\{I \ldots VII\}$
$(-2 \ldots 1,1)$		Φ	7	$\{I \ldots VII\}$
$(-2 \ldots -1,1)$		Φ	9	$\{I \ldots VII\}$
$(2,1)$	$F_\sharp$	Φ	5,9	$\{I \ldots VII\}$
$(1 \ldots 2,1)$		Φ	4	$\{I \ldots VII\}$
$(1,1)$	B	Φ	8	$\{I \ldots VII\}$
$(-1 \ldots 2,1)$		Φ	7	$\{I \ldots VII\}$
$(1 \ldots 2,-1)$		Φ	9	$\{I \ldots VII\}$
$(-2,-1)$	$G_\flat$	Φ	5,9	$\{I \ldots VII\}$
$(-2 \ldots -1,-1)$		Φ	4	$\{I \ldots VII\}$
$(-1,-1)$	$D_\flat$	Φ	8	$\{II, IV, VI\}$
$(0,2)$	$G_\sharp$	Φ	7	$\{II, IV, VI\}$
$(0,-2)$	$F_\flat$	Φ	7	$\{I \ldots VII\}$

P.2.6 Modulation Steps Between Melodic Minor Scales

We have the two modulators

$$\Phi_1 = e^b,$$
$$\Phi_2 = e^b.A.$$

The minimal cadence sets are these:

$$J_1 = \{I\}, J_2 = \{II\}, J_3 = \{III\},$$
$$J_4 = \{I, IV\}, J_5 = \{VI\}, J_6 = \{VII\}.$$

Pivots for the Modulations Between Melodic Minor Scales				
Translation	*Target Tonic*	*Modulator*	*Cadence*	*Pivots*
$(1, 0)$	G	Φ_2	6	$\{V, VII\}$
$(-1, 0)$	F	Φ_2	2	$\{II, IV\}$
$(2, 0)$	D	Φ_2	2	$\{II, IV\}$
$(2, 0)$	D	Φ_1	5	$\{II, IV, VII\}$
$(-2, 0)$	$B_\flat$	Φ_1	6	$\{V, VII\}$
$(-2, 0)$	$B_\flat$	Φ_1	5	$\{II, IV, VII\}$
$(0, 1)$	E	Φ_2	2	$\{II, V, VII\}$
$(0, 1)$	E	Φ_2	3	$\{I, III, IV, VI\}$
$(0, 1)$	E	Φ_1	5	$\{I, III, IV, VI\}$
$(0, 1)$	E	Φ_1	6	$\{II, III, V, VII\}$
$(0, -1)$	$A_\flat$	Φ_1	5	$\{I, III, IV, VI\}$
$(0, -1)$	$A_\flat$	Φ_2	5	$\{I, III, V, VI\}$
$(0, -1)$	$A_\flat$	Φ_1	6	$\{II, III, V, VII\}$
$(0, -1)$	$A_\flat$	Φ_2	6	$\{II, IV, VII\}$
$(-1, 1)$	A	Φ_1	2	$\{II, IV, VI, VII\}$
$(-1, 1)$	A	Φ_1	3	$\{I, III, V, VI\}$
$(-1, 1)$	A	Φ_2	5	$\{I, III, V, VI\}$
$(-1, 1)$	A	Φ_2	6	$\{II, IV, VII\}$
$(1, -1)$	$E_\flat$	Φ_1	2	$\{II, IV, VI, VII\}$
$(1, -1)$	$E_\flat$	Φ_1	3	$\{I, III, V, VI\}$
$(-1, 1)$	$E_\flat$	Φ_2	3	$\{I, III, IV, VI\}$
$(-1, 1)$	$E_\flat$	Φ_2	2	$\{II, V, VII\}$
$(-2, 1)$	D^*	Φ_2	5	$\{III, V, VII\}$
$(2, -1)$	$B_\flat^*$	Φ_2	1	$\{II, IV, VI\}$

P.2.7 General Modulation Behaviour for 32 Alterated Scales

The following list refers to the 32 scales as defined in 27.1.6.1. Following Radl [871], we say that a scale type

- *has no modulations* if its modulation domain is empty (always excluding the start tonality!),
- *has infinite modulations* if its modulation domain is infinite,
- *has modulations* if its modulation domain is not empty (always excluding the start tonality!), *has limited modulations* if the transitive closure (all tonics which can be reached by successive modulations from relative modulation domains) of its modulation domain is not total space.

Modulation Behaviour in 32 Altered Scale Types		
No.	*Scale Type*	*Behaviour*
1	c, d, e, f, g, a, b	has modulations: see table P.2.1
2	$c, d, e, f, g, a, b_\flat$	has modulations: $\pm(1,0)$ with III, $\pm(0,1)$ with V, VI
3	$c, d, e, f, g, a_\flat, b$	has modulations: corresponds to No. 11
4	$c, d, e, f, g, a_\flat, b_\flat$	has modulations: corresponds to No. 9
5	$c, d, e, f_\sharp, g, a, b$	has modulations: see table P.2.2
6	$c, d, e, f_\sharp, g, a, b_\flat$	has modulations (special table in [871])
7	$c, d, e, f_\sharp, g, a_\flat, b$	has modulations: $\pm(1,0)$ with II, $\pm(0,1)$ with III, VII
8	$c, d, e, f_\sharp, g, a_\flat, b_\flat$	has limited modulations: see No. 25
9	$c, d, e_\flat, f, g, a, b$	has modulations: see table P.2.6
10	$c, d, e_\flat, f, g, a, b_\flat$	has modulations: corresponds to No. 2
11	$c, d, e_\flat, f, g, a_\flat, b$	has modulations: see table P.2.5
12	$c, d, e_\flat, f, g, a_\flat, b_\flat$	has modulations: see table P.2.2
13	$c, d, e_\flat, f_\sharp, g, a, b$	has modulations: $\pm(1,0)$ with II, $\pm(-1,1)$ with I, VII
14	$c, d, e_\flat, f_\sharp, g, a, b_\flat$	has modulations: $\pm(1,1)$ with I, V, $\pm(0,1)$ with I, V
15	$c, d, e_\flat, f_\sharp, g, a_\flat, b$	has no modulations
16	$c, d, e_\flat, f_\sharp, g, a_\flat, b_\flat$	has modulations: corresponds to No. 7
17	$c, d_\flat, e, f, g, a, b$	has modulations: see No. 7
18	$c, d_\flat, e, f, g, a, b_\flat$	has modulations: corresponds to No. 14
19	$c, d_\flat, e, f, g, a_\flat, b$	has no modulations: see No. 15
20	$c, d_\flat, e, f, g, a_\flat, b_\flat$	has modulations: corresponds to No. 13
21	$c, d_\flat, e, f_\sharp, g, a, b$	has infinite modulations
22	$c, d_\flat, e, f_\sharp, g, a, b_\flat$	has infinite modulations
23	$c, d_\flat, e, f_\sharp, g, a_\flat, b$	has modulations: $\pm(1,0)$ with V, $\pm(0,1)$ with II
24	$c, d_\flat, e, f_\sharp, g, a_\flat, b_\flat$	has infinite modulations
25	$c, d_\flat, e_\flat, f, g, a, b$	has limited modulations
26	$c, d_\flat, e_\flat, f, g, a, b_\flat$	has modulations: corresponds to No. 6
27	$c, d_\flat, e_\flat, f, g, a_\flat, b$	has modulations: see No. 16
28	$c, d_\flat, e_\flat, f, g, a_\flat, b_\flat$	has modulations: see table P.2.1
29	$c, d_\flat, e_\flat, f_\sharp, g, a, b$	has infinite modulations: corresponds to No. 24
30	$c, d_\flat, e_\flat, f_\sharp, g, a, b_\flat$	has infinite modulations: corresponds to No. 22
31	$c, d_\flat, e_\flat, f_\sharp, g, a_\flat, b$	has modulations: corresponds to No. 23
32	$c, d_\flat, e_\flat, f_\sharp, g, a_\flat, b_\flat$	has infinite modulations: corresponds to No. 21

Q

Counterpoint Steps

Q.1 Contrapuntal Symmetries

All the following tables relate to representatives of strong dichotomies (X/Y) which are indicated in the table after the counterpoint theorem 33 in Section 31.3.3.

Q.1.1 Class No. 64

k	g	$g.X[\varepsilon]$	$g.X[\varepsilon] \cap X[\varepsilon]$	$card(g.X[\varepsilon] \cap X[\varepsilon])$
2	$e^6(1+\varepsilon.6)$	$(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.e^6X$	z even: $z + \varepsilon.\{5,11\}$ z odd: $z + \varepsilon.X$	48
	$e^{\varepsilon.8}5$	$\mathbb{Z}_{12} + \varepsilon.e^8 5X$	$\mathbb{Z}_{12} + \varepsilon.\{4,5,7,9\}$	
4	$e^6(1+\varepsilon.6)$	see $k=2$	see $k=2$	48
5	$e^{\varepsilon.11}11$	$\mathbb{Z}_{12} + \varepsilon.e^{11}11X$	$\mathbb{Z}_{12} + \varepsilon.\{2,4,7,9\}$	48
7	$e^6(1+\varepsilon.6)$	see $k=2$	see $k=2$	48
9	$e^6(1+\varepsilon.6)$	see $k=2$	see $k=2$	48
	$e^{\varepsilon.3}7$	$\mathbb{Z}_{12} + \varepsilon.e^3 7X$	$\mathbb{Z}_{12} + \varepsilon.\{2,4,5,7\}$	
11	$e^{\varepsilon.3}7$	see $k=9$	see $k=9$	48
	$e^{\varepsilon.11}11$	see $k=5$	see $k=5$	
	$e^{\varepsilon.8}5$	see $k=2$	see $k=2$	

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Q.1.2 Class No. 68

k	g	$g.X[\varepsilon]$	$g.X[\varepsilon] \cap X[\varepsilon]$	$card(g.X[\varepsilon] \cap X[\varepsilon])$
0	$e^{\varepsilon.6}7$	$\mathbb{Z}_{12} + \varepsilon.e^6 7X$	$\mathbb{Z}_{12} + \varepsilon.(X - \{0\})$	60
1	$e^{\varepsilon.3}7$	$\mathbb{Z}_{12} + \varepsilon.e^3 7X$	$\mathbb{Z}_{12} + \varepsilon.\{0,2,3,5\}$	48
	$e^{\varepsilon.6}(1+\varepsilon.6)$	$(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.e^6 X$	z even: $z + \varepsilon.\{2,8\}$	
			z odd: $z + \varepsilon.X$	
	$7 + \varepsilon.6$	$(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.7X$	z even: $z + \varepsilon.\{0,2,8\}$	
			z odd: $z + \varepsilon.(X - \{0\})$	
2	5	$\mathbb{Z}_{12} + \varepsilon.5X$	$\mathbb{Z}_{12} + \varepsilon.\{0,1,3,5\}$	48
3	$e^{\varepsilon.6}(1+\varepsilon.6)$	$(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.e^6 X$	z even: $z + \varepsilon.\{2,8\}$	48
			z odd: $z + \varepsilon.X$	
	$7 + \varepsilon.6$	$(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.7X$	z even: $z + \varepsilon.\{0,2,8\}$	
			z odd: $z + \varepsilon.(X - \{0\})$	
5	$e^{\varepsilon.3}11$	$\mathbb{Z}_{12} + \varepsilon.e^3 X$	$\mathbb{Z}_{12} + \varepsilon.\{0,1,2,3\}$	48
	$e^{\varepsilon.6}(1+\varepsilon.6)$	see $k = 1$	see $k = 1$	
	$7 + \varepsilon.6$	see $k = 3$	see $k = 3$	
8	$e^{\varepsilon.3}11$	see $k = 5$	see $k = 5$	48
	$e^{\varepsilon.3}7$	see $k = 1$	see $k = 1$	
	5	see $k = 2$	see $k = 2$	

Q.1.3 Class No. 71

k	g	$g.X[\varepsilon]$	$g.X[\varepsilon] \cap X[\varepsilon]$	$card(g.X[\varepsilon] \cap X[\varepsilon])$
0	$e^{\varepsilon.8}5$	$\mathbb{Z}_{12} + \varepsilon.e^8 5X$	$\mathbb{Z}_{12} + \varepsilon.\{1,2,6,7\}$	48
	$e^{\varepsilon.8}(5+\varepsilon.6)$	$(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.e^8 5X$	z even: $z + \varepsilon.\{1,2,6,7\}$	
			z odd: $z + \varepsilon.\{0,1,2,7\}$	
1	$e^{\varepsilon.3}(7+\varepsilon.3)$	$(1+\varepsilon.3)\mathbb{Z}_{12} + \varepsilon.e^3 7X$	$z = 0,4,8 : z + \varepsilon.\{0,3\}$	42
			$z = 1,5,9 : z + \varepsilon.\{0,1,3,6,7\}$	
			$z = 2,6,10 : z + \varepsilon.\{3,6\}$	
			$z = 3,7,11 : z + \varepsilon.\{0,1,2,6,7\}$	
	$e^{\varepsilon.3}(7-\varepsilon.3)$	$(1-\varepsilon.3)\mathbb{Z}_{12} + \varepsilon.e^3 7X$	$z = 0,4,8 : z + \varepsilon.\{0,3\}$	
			$z = 1,5,9 : z + \varepsilon.\{0,1,2,6,7\}$	
			$z = 2,6,10 : z + \varepsilon.\{3,6\}$	
			$z = 3,7,11 : z + \varepsilon.\{0,1,3,6,7\}$	
	$e^{\varepsilon.9}(7+\varepsilon.3)$	$(1+\varepsilon.3)\mathbb{Z}_{12} + \varepsilon.e^9 7X$	$z = 0,4,8 : z + \varepsilon.\{3,6\}$	
			$z = 1,5,9 : z + \varepsilon.\{0,1,2,6,7\}$	
			$z = 2,6,10 : z + \varepsilon.\{0,3\}$	
			$z = 3,7,11 : z + \varepsilon.\{0,1,3,6,7\}$	
	$e^{\varepsilon.9}(7-\varepsilon.3)$	$(1-\varepsilon.3)\mathbb{Z}_{12} + \varepsilon.e^9 7X$	$z = 0,4,8 : z + \varepsilon.\{3,6\}$	
			$z = 1,5,9 : z + \varepsilon.\{0,1,3,6,7\}$	
			$z = 2,6,10 : z + \varepsilon.\{0,3\}$	
			$z = 3,7,11 : z + \varepsilon.\{0,1,2,6,7\}$	
2	$e^{\varepsilon.6}(1+\varepsilon.6)$	$(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.e^6 X$	z even: $z + \varepsilon.\{0,1,6,7\}$	60
			z odd: $z + \varepsilon.X$	
3	$e^{\varepsilon.6}(1+\varepsilon.6)$	see $k = 2$	see $k = 2$	60
6	$e^{\varepsilon.2}5$	$\mathbb{Z}_{12} + \varepsilon.e^2 5X$	$\mathbb{Z}_{12} + \varepsilon.\{0,1,2,7\}$	48
	$e^{\varepsilon.2}(5+\varepsilon.6)$	$(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.e^2 5X$	z even: $z + \varepsilon.\{0,1,2,7\}$	
			z odd: $z + \varepsilon.\{1,2,6,7\}$	
7	$e^{\varepsilon.9}(7+\varepsilon.3)$	see $k = 1$	see $k = 1$	42
	$e^{\varepsilon.3}(7+\varepsilon.3)$	see $k = 1$	see $k = 1$	

Q.1.4 Class No. 75

k	g	$g.X[\varepsilon]$	$g.X[\varepsilon] \cap X[\varepsilon]$	$card(g.X[\varepsilon] \cap X[\varepsilon])$
0	$e^{\varepsilon.9}7$	$\mathbb{Z}_{12} + \varepsilon.e^9 7X$	$\mathbb{Z}_{12} + \varepsilon.\{1,4,5,8\}$	48
	$e^{\varepsilon.9}(7+\varepsilon.4)$	$(1+\varepsilon.4)\mathbb{Z}_{12} + \varepsilon.e^9 7X$	$z = 0,3,6,9 : z + \varepsilon.\{1,4,5,8\}$	
			$z = 1,4,7,10 : z + \varepsilon.\{0,1,5,8\}$	
			$z = 2,5,8,11 : z + \varepsilon.\{0,1,4,5\}$	
	$e^{\varepsilon.9}(7-\varepsilon.4)$	$(1-\varepsilon.4)\mathbb{Z}_{12} + \varepsilon.e^9 7X$	$z = 0,3,6,9 : z + \varepsilon.\{1,4,5,8\}$	
			$z = 1,4,7,10 : z + \varepsilon.\{0,1,4,5\}$	
			$z = 2,5,8,11 : z + \varepsilon.\{0,1,5,8\}$	
	$e^{\varepsilon.6}(1+\varepsilon.6)$	$(1+\varepsilon.6)\mathbb{Z}_{12} + \varepsilon.e^6 X$	z even: $z + \varepsilon.\{2,8\}$	
			z odd: $z + \varepsilon.X$	
1	$e^{\varepsilon.6}(1+\varepsilon.6)$	see $k = 0$	see $k = 0$	48
2	$e^{\varepsilon.8}(5+\varepsilon.4)$	$(1+\varepsilon.4)\mathbb{Z}_{12} + \varepsilon.e^8 5X$	$z = 0,3,6,9 : z + \varepsilon.\{0,1,4,8\}$	56
			$z = 1,4,7,10 : z + \varepsilon.\{0,1,4,5,8\}$	
			$z = 2,5,8,11 : z + \varepsilon.\{0,2,4,5,8\}$	
	$e^{\varepsilon.8}(5-\varepsilon.4)$	$(1-\varepsilon.4)\mathbb{Z}_{12} + \varepsilon.e^8 5X$	$z = 0,3,6,9 : z + \varepsilon.\{0,1,4,8\}$	
			$z = 1,4,7,10 : z + \varepsilon.\{0,2,4,5,8\}$	
			$z = 2,5,8,11 : z + \varepsilon.\{0,1,4,5,8\}$	
4	$e^{\varepsilon.6}(1+\varepsilon.6)$	see $k = 1$	see $k = 1$	48
5	$e^{\varepsilon.8}(1 \pm \varepsilon.4)$	see $k = 2$	see $k = 2$	56
8	$e^{\varepsilon.5}11$	$\mathbb{Z}_{12} + \varepsilon.e^5 11X$	$\mathbb{Z}_{12} + \varepsilon.\{0,1,4,5\}$	48
	$e^{\varepsilon.5}(11+\varepsilon.4)$	$(1+\varepsilon.4)\mathbb{Z}_{12} + \varepsilon.e^5 11X$	$z = 0,3,6,9 : z + \varepsilon.\{0,1,4,5\}$	
			$z = 1,4,7,10 : z + \varepsilon.\{1,4,5,8\}$	
			$z = 2,5,8,11 : z + \varepsilon.\{0,1,5,8\}$	
	$e^{\varepsilon.5}(11-\varepsilon.4)$	$(1-\varepsilon.4)\mathbb{Z}_{12} + \varepsilon.e^5 11X$	$z = 0,3,6,9 : z + \varepsilon.\{0,1,4,5\}$	
			$z = 1,4,7,10 : z + \varepsilon.\{0,1,5,8\}$	
			$z = 2,5,8,11 : z + \varepsilon.\{1,4,5,8\}$	

Q.1.5 Class No. 78

k	g	$g.X[\varepsilon]$	$g.X[\varepsilon] \cap X[\varepsilon]$	$card(g.X[\varepsilon] \cap X[\varepsilon])$
0	$e^{\varepsilon.9}(7+\varepsilon.3)$	$(1+\varepsilon.3)\mathbb{Z}_{12}+\varepsilon.e^{9}7X$	$z=0,4,8 : z+\varepsilon.\{1,4\}$	42
			$z=1,5,9 : z+\varepsilon.\{0,2,4,6,10\}$	
			$z=2,6,10 : z+\varepsilon.\{1,10\}$	
			$z=3,7,11 : z+\varepsilon.\{0,1,4,6,10\}$	
	$e^{\varepsilon.9}(7-\varepsilon.3)$	$(1-\varepsilon.3)\mathbb{Z}_{12}+\varepsilon.e^{9}7X$	$z=0,4,8 : z+\varepsilon.\{1,4\}$	
			$z=1,5,9 : z+\varepsilon.\{0,1,4,6,10\}$	
			$z=2,6,10 : z+\varepsilon.\{1,10\}$	
			$z=3,7,11 : z+\varepsilon.\{0,2,4,6,10\}$	
	$e^{\varepsilon.3}(7+\varepsilon.3)$	$(1+\varepsilon.3)\mathbb{Z}_{12}+\varepsilon.e^{3}7X$	$z=0,4,8 : z+\varepsilon.\{1,10\}$	
			$z=1,5,9 : z+\varepsilon.\{0,1,4,6,10\}$	
			$z=2,6,10 : z+\varepsilon.\{1,4\}$	
			$z=3,7,11 : z+\varepsilon.\{0,2,4,6,10\}$	
	$e^{\varepsilon.3}(7-\varepsilon.3)$	$(1-\varepsilon.3)\mathbb{Z}_{12}+\varepsilon.e^{3}7X$	$z=0,4,8 : z+\varepsilon.\{1,10\}$	
			$z=1,5,9 : z+\varepsilon.\{0,2,4,6,10\}$	
			$z=2,6,10 : z+\varepsilon.\{1,4\}$	
			$z=3,7,11 : z+\varepsilon.\{0,1,4,6,10\}$	
1	$e^{\varepsilon.6}(1+\varepsilon.6)$	$(1+\varepsilon.6)\mathbb{Z}_{12}+\varepsilon.e^{6}X$	z even: $z+\varepsilon.\{0,4,6,10\}$	60
			z odd: $z+\varepsilon.X$	
2	$e^{\varepsilon.6}(1+\varepsilon.6)$	see $k=1$	see $k=1$	60
4	$5+\varepsilon.4$	$(1+\varepsilon.4)\mathbb{Z}_{12}+\varepsilon.5X$	$z=0,3,6,9 : z+\varepsilon.\{0,2,6,10\}$	56
			$z=1,4,7,10 : z+\varepsilon.\{0,2,4,6,10\}$	
			$z=2,5,8,11 : z+\varepsilon.\{1,2,4,6,10\}$	
	$5-\varepsilon.4$	$(1-\varepsilon.4)\mathbb{Z}_{12}+\varepsilon.5X$	$z=0,3,6,9 : z+\varepsilon.\{0,2,6,10\}$	
			$z=1,4,7,10 : z+\varepsilon.\{1,2,4,6,10\}$	
			$z=2,5,8,11 : z+\varepsilon.\{0,2,4,6,10\}$	
6	$e^{\varepsilon.9}(7\pm\varepsilon.3)$	see $k=0$	see $k=0$	42
	$e^{\varepsilon.3}(7\pm\varepsilon.3)$	see $k=0$	see $k=0$	
10	$e^{\varepsilon.6}(5+\varepsilon.2)$	$(1+\varepsilon.2)\mathbb{Z}_{12}+\varepsilon.e^{6}5X$	$z=0,6 : z+\varepsilon.\{0,2,4,6\}$	52
			$z=1,7 : z+\varepsilon.\{1,2,4,6,10\}$	
			$z=2,8 : z+\varepsilon.\{0,4,6,10\}$	
			$z=3,9 : z+\varepsilon.\{0,2,6,10\}$	
			$z=4,10 : z+\varepsilon.\{0,2,4,10\}$	
			$z=5,11 : z+\varepsilon.\{0,2,4,6,10\}$	
	$e^{\varepsilon.6}(5-\varepsilon.2)$	$(1-\varepsilon.2)\mathbb{Z}_{12}+\varepsilon.e^{6}5X$	$z=0,6 : z+\varepsilon.\{0,2,4,6\}$	
			$z=1,7 : z+\varepsilon.\{0,2,4,6,10\}$	
			$z=2,8 : z+\varepsilon.\{0,2,4,10\}$	
			$z=3,9 : z+\varepsilon.\{0,2,6,10\}$	
			$z=4,10 : z+\varepsilon.\{0,4,6,10\}$	
			$z=5,11 : z+\varepsilon.\{1,2,4,6,10\}$	

Q.1.6 Class No. 82

k	g	$g.X[\varepsilon]$	$g.X[\varepsilon] \cap X[\varepsilon]$	$card(g.X[\varepsilon] \cap X[\varepsilon])$
0	$e^{\varepsilon.6}(1+\varepsilon.6)$	$(1+\varepsilon.6)\mathbb{Z}_{12}+\varepsilon.e^{6}X$	z even: $z+\varepsilon.\{3,9\}$	48
			z odd: $z+\varepsilon.X$	
	$e^{\varepsilon.6}(7+\varepsilon.6)$	$(1+\varepsilon.6)\mathbb{Z}_{12}+\varepsilon.e^{6}7X$	z even: $z+\varepsilon.\{3,7,9\}$	
			z odd: $z+\varepsilon.(X-\{7\})$	
	$e^{\varepsilon.11}(11-\varepsilon.4)$	$(1+\varepsilon.4)\mathbb{Z}_{12}+\varepsilon.e^{11}11X$	$z=0,3,6,9: z+\varepsilon.\{3,4,7,8\}$	
			$z=1,4,7,10: z+\varepsilon.\{0,3,7,8\}$	
			$z=2,5,8,11: z+\varepsilon.\{0,3,4,7\}$	
	$e^{\varepsilon.11}(11+\varepsilon.4)$	$(1-\varepsilon.4)\mathbb{Z}_{12}+\varepsilon.e^{11}11X$	$z=0,3,6,9: z+\varepsilon.\{3,4,7,8\}$	
			$z=1,4,7,10: z+\varepsilon.\{0,3,4,7\}$	
			$z=2,5,8,11: z+\varepsilon.\{0,3,7,8\}$	
	$e^{\varepsilon.11}11$	$\mathbb{Z}_{12}+\varepsilon.e^{11}11X$	$\mathbb{Z}_{12}+\varepsilon.\{3,4,7,8\}$	
3&9	$e^{\varepsilon.8}(5-\varepsilon.4)$	$(1+\varepsilon.4)\mathbb{Z}_{12}+\varepsilon.e^{8}5X$	$z=0,3,6,9: z+\varepsilon.\{0,4,7,8\}$	56
			$z=1,4,7,10: z+\varepsilon.(X-\{7\})$	
			$z=2,5,8,11: z+\varepsilon.(X-\{9\})$	
	$e^{\varepsilon.8}(5+\varepsilon.4)$	$(1-\varepsilon.4)\mathbb{Z}_{12}+\varepsilon.e^{8}5X$	$z=0,3,6,9: z+\varepsilon.\{0,4,7,8\}$	
			$z=1,4,7,10: z+\varepsilon.(X-\{9\})$	
			$z=2,5,8,11: z+\varepsilon.(X-\{7\})$	
4	$e^{\varepsilon.6}(1+\varepsilon.6)$	$(1+\varepsilon.6)\mathbb{Z}_{12}+\varepsilon.e^{6}X$	see $k=0$	48
	$e^{\varepsilon.6}(7+\varepsilon.6)$	$(1+\varepsilon.6)\mathbb{Z}_{12}+\varepsilon.e^{6}7X$	see $k=0$	
7	7	$\mathbb{Z}_{12}+\varepsilon.7X$	$\mathbb{Z}_{12}+\varepsilon.(X-\{7\})$	60
8	$e^{\varepsilon.3}7$	$\mathbb{Z}_{12}+\varepsilon.e^{3}7X$	$\mathbb{Z}_{12}+\varepsilon.\{0,3,4,7\}$	48
	$e^{\varepsilon.6}(1+\varepsilon.6)$	$(1+\varepsilon.6)\mathbb{Z}_{12}+\varepsilon.e^{6}X$	see $k=0$	
	$e^{\varepsilon.6}(7+\varepsilon.6)$	$(1+\varepsilon.6)\mathbb{Z}_{12}+\varepsilon.e^{6}7X$	see $k=0$	
	$e^{\varepsilon.3}(7+\varepsilon.4)$	$(1+\varepsilon.4)\mathbb{Z}_{12}+\varepsilon.e^{3}7X$	$z=0,3,6,9: z+\varepsilon.\{0,3,4,7\}$	
			$z=1,4,7,10: z+\varepsilon.\{3,4,7,8\}$	
			$z=2,5,8,11: z+\varepsilon.\{0,3,7,8\}$	
	$e^{\varepsilon.3}(7-\varepsilon.4)$	$(1-\varepsilon.4)\mathbb{Z}_{12}+\varepsilon.e^{3}7X$	$z=0,3,6,9: z+\varepsilon.\{0,3,4,7\}$	
			$z=1,4,7,10: z+\varepsilon.\{0,3,7,8\}$	
			$z=2,5,8,11: z+\varepsilon.\{3,4,7,8\}$	

Q.2 Permitted Successors for the Major Scale

For the sweeping orientation, given a cantus firmus step $CF : x \mapsto y$, one is allowed to move from a consonance c (i.e., $x + \varepsilon.c$) in the top row to a consonance d (i.e., $y + \varepsilon.d$) in the right column iff there is a $*$ in the corresponding matrix entry.

1. Oblique Motion in Cantus Firmus						
$CF : 0 \mapsto 0$	$CF : 2 \mapsto 2$	$CF : 4 \mapsto 4$	$CF : 5 \mapsto 5$	$CF : 7 \mapsto 7$	$CF : 9 \mapsto 9$	$CF : 11 \mapsto 11$

0 4 7 9
  * * 0
* * 4
* * 7
* * * 9
	0 3 7 9
	  * * * 0
	* * 3
	* * 7
	* * 9
		0 3 7 8
		  * * * 0
		* * * 3
		* * 7
		* * * 8
			0 4 7 9
			  * * 0
			* * 4
			* * 7
			* * * 9
				0 4 7 9
				  * * 0
				* * 4
				* * 7
				* * * 9
					0 3 7 8
					  * * * 0
					* * * 3
					* * 7
					* * * 8
						0 3 8
						  * * 0
						* * 3
						* * 8

2. Minor Ascending Second in Cantus Firmus						
$CF : 4 \mapsto 5$	$CF : 11 \mapsto 0$					
0 3 7 8 * * * * 0 * * * * 4 * * * 7 * * * * 9	0 3 8 * * * 0 * * * 4 * * * 7 * * * 9					

3. Minor Descending Second in Cantus Firmus						
$CF : 5 \mapsto 4$	$CF : 0 \mapsto 11$					
0 4 7 9 * * * * 0 * * * * 3 * * 7 * * * * 8	0 4 7 9 * * * * 0 * * * * 3 * * * * 8					

4. Major Ascending Second in Cantus Firmus						
$CF : 0 \mapsto 2$	$CF : 2 \mapsto 4$	$CF : 5 \mapsto 7$	$CF : 7 \mapsto 9$	$CF : 9 \mapsto 11$		
0 4 7 9 * * * 0 * * * * 3 * * 7 * * * * 9	0 3 7 9 * * * * 0 * * * * 3 * * 7 * * * * 8	0 4 7 8 * * * 0 * * * 4 * * 7 * * * * 9	0 4 7 9 * * * 0 * * * * 3 * * 7 * * * 8	0 3 7 8 * * * * 0 * * * * 3 * * * * 8		

5. Major Descending Second in Cantus Firmus						
$CF : 2 \mapsto 0$	$CF : 4 \mapsto 2$	$CF : 7 \mapsto 5$	$CF : 9 \mapsto 7$	$CF : 11 \mapsto 9$		
0 3 7 9 * * * 0 * * * * 4 * * 7 * * * * 9	0 3 7 8 * * * * 0 * * * * 3 * * 7 * * * * 9	0 4 7 9 * * * 0 * * * 4 * * 7 * * * * 9	0 3 7 8 * * * 0 * * * * 4 * * 7 * * * 9	0 3 8 * * * 0 * * * 3 * * * 7 * * * 8		

6. Minor Ascending Third in Cantus Firmus					
$CF : 2 \mapsto 5$	$CF : 4 \mapsto 7$	$CF : 9 \mapsto 0$	$CF : 11 \mapsto 2$		
0 3 7 9 * * 0 * * * * 4 * * * 7 * * 9	0 3 7 8 * * * * 0 * * * * 4 * * * 7 * * * 9	0 3 7 8 * * 0 * * * 4 * * * 7 * * * 9	0 3 8 * * * 0 * * 3 * * * 7 * * 9		

7. Minor Descending Third in Cantus Firmus					
$CF : 5 \mapsto 2$	$CF : 7 \mapsto 4$	$CF : 0 \mapsto 9$	$CF : 2 \mapsto 11$		
0 4 7 9 * * * * 0 * * * 3 * * * 7 * * * 9	0 4 7 9 * * * * 0 * * * 3 * * * 7 * * * * 8	0 4 7 9 * * * * 0 * * * 3 * * * 7 * * * * 8	0 3 7 9 * * * * 0 * * * * 3 * * * * 8		

8. Major Ascending Third in Cantus Firmus					
$CF : 0 \mapsto 4$	$CF : 5 \mapsto 9$	$CF : 7 \mapsto 11$			
0 4 7 9 * * 0 * * * * 3 * * * 7 * * * 8	0 4 7 9 * * 0 * * * * 3 * * * 7 * * * 8	0 4 7 9 * * 0 * * * * 3 * * * 8			

9. Major Descending Third in Cantus Firmus					
$CF : 4 \mapsto 0$	$CF : 9 \mapsto 5$	$CF : 11 \mapsto 7$			
0 3 7 8 * * * * 0 * * * * 4 * * * 7 * * * * 9	0 3 7 8 * * * * 0 * * * * 4 * * * 7 * * * * 9	0 3 8 * * * 0 * * * 4 * * * 7 * * * 9			

10. Ascending Fourth in Cantus Firmus					
$CF : 0 \mapsto 5$	$CF : 2 \mapsto 7$	$CF : 4 \mapsto 9$	$CF : 7 \mapsto 0$	$CF : 9 \mapsto 2$	$CF : 11 \mapsto 4$
0 4 7 9 * * * * 0 * * * * 4 * * * 7 * * * * 9	0 3 7 9 * * * * 0 * * * * 4 * * * 7 * * * * 9	0 3 7 8 * * * * 0 * * * * 3 * * * 7 * * * * 8	0 4 7 9 * * * * 0 * * * * 4 * * * 7 * * * * 9	0 3 7 8 * * * * 0 * * * * 3 * * * 7 * * * * 8	0 3 8 * * * 0 * * * 3 * * * 7 * * * 8

11. Descending Fourth in Cantus Firmus					
$CF : 5 \mapsto 0$	$CF : 7 \mapsto 2$	$CF : 9 \mapsto 4$	$CF : 0 \mapsto 7$	$CF : 2 \mapsto 9$	$CF : 4 \mapsto 11$
0 4 7 9 * * * * 0 * * * * 4 * * * 7 * * * * 9	0 4 7 9 * * * * 0 * * * * 3 * * * 7 * * * * 9	0 3 7 8 * * * * 0 * * * * 3 * * * 7 * * * * 8	0 4 7 9 * * * * 0 * * * * 4 * * * 7 * * * * 9	0 3 7 9 * * * * 0 * * * * 3 * * * 7 * * * * 8	0 3 7 8 * * * * 0 * * * * 3 * * * * 8

12. Ascending Tritone in Cantus Firmus					
$CF : 5 \mapsto 11$	$CF : 11 \mapsto 5$				
0 4 7 9 * * 0 * * * 3 * * * 8	0 3 8 * * 0 * * * 4 * * * 7 * * 9				

Part XXIV

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